Bell’s inequality without alternative settings

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Abstract
A suitable generalized measurement described by a 4-element positive operator-valued measure (POVM) on each particle of a two-qubit system in the singlet state is, from the point of view of Einstein, Podolsky, and Rosen’s (EPR’s) criterion of elements of reality, equivalent to a random selection between two alternative projective measurements. It is shown that an EPR-experiment with a fixed POVM on each particle provides a violation of Bell’s inequality without requiring local observers to choose between the alternatives. This approach could be useful for designing a loophole-free test of Bell’s inequality.

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1. Introduction
1.1. EPR’s elements of reality with projective measurements

Bell [1] discovered that some predictions of quantum mechanics contradict Einstein, Podolsky, and Rosen’s (EPR’s) “elements of reality” [2], defined as those satisfying the following criterion: “If, without in any way disturbing a system, we can predict with certainty (i.e., with probability equal to unity) the value of a physical quantity, then there exists an element of physical reality corresponding to this physical quantity.” In Bohm’s version of the EPR’s experiment [3], when two space-like separated projective measurements\(^{2}\) of the spin along the same direction \(\vec{n}\) are performed on both spin-1/2 particles prepared in the singlet state\(^{3}\)

\[
|\psi\rangle = \frac{1}{\sqrt{2}} (|n = +1, n = -1\rangle - |n = -1, n = +1\rangle),
\]

measurements have pre-determined results, except for those which satisfy the criterion of elements of reality. From this point of view, EPR theories are just a subset of local hidden-variable theories.

\(^{2}\) Projective measurements are those represented in quantum mechanics by self-adjoint operators \(O = \sum_r rP_r\), where \(P_r\) is the projector onto the eigenspace of \(O\) with eigenvalue \(r\). The possible outcomes of the measurement correspond to the eigenvalues \(r\).

\(^{3}\) \(|n = +1, n = -1\rangle\) represents the quantum state in which, when the spin along \(\vec{n}\) is measured on both particles, we obtain the result +1 for particle 1, and −1 for particle 2.

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quantum theory predicts that the results (+1 or −1) will be opposite for any \( \vec{n} \). Therefore, there is an element of reality corresponding to every spin component of either of the two particles, since an observer performing a measurement of the spin along \( \vec{n} \) on particle 1 can predict with certainty the result of a measurement of the spin along \( \vec{n} \) on particle 2 without disturbing it (since the particles are distant enough to exclude communication at the speed of light or lower). In other words, according to EPR, each particle must contain a set of instructions [4] which determines the result of any spin measurement.

1.2. Bell’s inequality with projective measurements

In the most common Bell’s inequality, the Clauser–Horne–Shimony–Holt (CHSH) inequality [5], two alternative dichotomic (i.e., with possible results +1 or −1) observables \( A \) or \( a \) are measured on particle 1, and other two, \( B \) or \( b \), on particle 2. For instance, these observables could be the spin along \( \vec{A} \) or \( \vec{a} \) on particle 1 and along \( \vec{B} \) or \( \vec{b} \) on particle 2. If the results of \( A, a, B, b \) are predefined for all pairs of particles, these results must satisfy the CHSH inequality:

\[
|\langle AB - Ab - aB - ab \rangle| \leq 2,
\]

where \( \langle \rangle \) means average over all pairs. However, in quantum mechanics two spin components of the same spin-1/2 particle, like \( A \) and \( a \) (or \( B \) and \( b \)), cannot be measured in the same experiment. Thus the quantum equivalent of the left-hand side in inequality (2) is usually expressed as

\[
\hat{B} = |\langle AB \rangle_{\psi} - \langle Ab \rangle_{\psi} - \langle aB \rangle_{\psi} - \langle ab \rangle_{\psi}|,
\]

where \( \langle AB \rangle_{\psi} \) is the mean value of the product of the results of measuring \( A \) on particle 1 and \( B \) on particle 2. It is a common assumption that any test of the CHSH inequality requires two local observers who have free-will to choose between two possible settings during the flight of the particles and thus performing four different experiments (\( AB, Ab, aB, ab \)) on four different subensembles of pairs (Fig. 1). Quantum mechanics predict that, for pairs prepared in the singlet state (1), for certain choices of \( \vec{A}, \vec{a}, \vec{B}, \) and \( \vec{b} \), we will obtain \( \hat{B} > 2 \), which violates the CHSH inequality (2). A widely extended belief is that a setup with fixed local measurements cannot be used to test the CHSH inequality.

In Section 2, I will show that a suitable generalized measurement is, from EPR’s criterion of elements of reality, equivalent to a random selection between two alternative projective measurements. In Section 3, I will show that a violation of the CHSH inequality (2) can be obtained without requiring local observers to choose between alternative projective measurements.

1.3. The locality and detection loopholes

Experiments to test Bell’s inequality [6–16] have agreed with quantum predictions and seem to exclude elements of reality. However, up until now, all performed experiments are subject to at least one of two loopholes.

The locality loophole [17,18] arises whenever measurements performed on two spatially separated particles are not space-like separated and thus the possibility of communication at the speed of light between the two parts cannot be excluded. The detection loophole [19,20] arises from the fact that in most experiments only a small subset of all the created pairs are actually detected. It is therefore necessary to assume that the registered pairs are a fair sample of all the emitted pairs (fair sampling assumption). In practice, both loopholes are not independent [21]. These loopholes are natural from the local hidden variables’ point of view, in which particles have additional hidden vari-
ables that enable them to give results for certain experiments and not for others (for instance, to pass an analyzer for certain settings and not for others). If the actual setting does not correspond to the hidden variables of the particle, then, according to the detection loophole, the particle is not detected. Whereas, according to the locality loophole, this situation never happens because the particle knows the setting in advance.

The experiment in [15] with polarization-entangled photons and a 400 m separation between the particles (which gives the observer 1.3 μs to make the selection-measurement process, defined in [15] “to last from the first point in time which can influence the choice of the analyzer setting until the final registration of the photon”) is not subject to the locality loophole, but the detection efficiency (5%) is not high enough to close the detection loophole (82.8% would be required [22]).

On the other hand, the experiment in [16] with trapped beryllium ions has nearly perfect detection efficiency and thereby is not subject to the detection loophole, but the distance between the ions (3 μm), although large enough that no known interaction could affect the results, is not large enough to close the locality loophole, because the selection-measurement requires two steps: a selection (equivalent to rotating a wave-plate in the case of experiments with polarized photons) applying Raman beams for a pulse of a duration of 400 ns, and a measurement probing the ion with circularly polarized light from a “detector” laser beam during this detection pulse; if the ion is in one state, it scatters many photons; if it is in the orthogonal state, it scatters very few photons.

It was first thought that improving the detection efficiency in experiments with pairs of entangled photons would avoid both loopholes, but this proved more difficult than expected and, despite several proposals [23,24], no conclusive experiment has been achieved. Another approach based on pairs of atoms produced through a photodissociation process [25,26], or pairs of Rydberg atoms [27], is not easy to implement and no conclusive test of Bell’s inequality has been carried out in these systems [28].

Summing up, although the recent experiments [15, 16] have meant a significant advance, they still have not settled the debate [29]. A loophole-free experiment is still demanded [29,30].

In Section 4 we will show that the approach to test Bell’s inequality introduced in Sections 2 and 3 could be useful to reduce the distance requirements to close the locality loophole in experiments with a high enough detection efficiency.

2. EPR’s elements of reality with POVMs

Motivated by the quantum information approach to quantum mechanics and by the fact that current technology allows an exquisite level of control over the measurements that can be performed, recent formulations of the principles of quantum mechanics [31,32] stress that the measurements correspond to positive operator-valued measures (POVMs) [31–34], extending the notion of von Neumann’s projection-valued measures. The main difference between POVMs and von Neumann’s projection valued measures is that for POVMs the number of available outcomes of a measurement may be higher than the dimensionality of the Hilbert space. An N-outcome generalized measurement is represented by an N-element POVM which consists of N positive-semidefinite operators \( \{ E_d \} \) that sum the identity (i.e., \( \sum_d E_d = 1 \)). Neumark’s theorem [37] guarantees that there always exists a realizable experimental procedure to generate any desired POVM. Any generalized measurement represented by a POVM can be seen as a von Neumann’s measurement on a larger Hilbert space. Therefore, any generalized measurement on a single qubit can be seen as a von Neumann’s joint measurement on a system composed by the qubit plus an auxiliary quantum system (ancilla).

A natural question is what POVMs means from the point of view of EPR’s elements of reality. To answer that, let us go back to the EPR–Bohm experiment. Let us suppose that, instead of the same projective measurement on both particles, a projective measurement \( A \) is performed on particle 1 and a 4-outcome generalized measurement is performed on particle 2 (Fig. 2). Specifically, let us consider the generalized measurement represented by the following 4-element POVM:

\[
E_{A+} = \frac{1}{2} \left( P_{A=+1} - P_{A=-1} \right),
\]

\[
= \frac{1}{2} |A = +1 \rangle \langle A = +1|,
\]

(4)

In Section 4 we will show that the approach to test Bell’s inequality introduced in Sections 2 and 3 could be useful to reduce the distance requirements to close the locality loophole in experiments with a high enough detection efficiency.
$$E_A = \frac{1}{2} P_{|A=+1\rangle} = \frac{1}{2} (\mathbb{1} - P_{|A=+1\rangle}),$$

$$= \frac{1}{2} |A = -1\rangle \langle A = -1|,$$  

(5)

$$E_{a+} = \frac{1}{2} P_{|a=-1\rangle} = \frac{1}{2} (\mathbb{1} - P_{|a=-1\rangle}),$$

$$= \frac{1}{2} |a = +1\rangle \langle a = +1|,$$  

(6)

$$E_{a-} = \frac{1}{2} P_{|a=+1\rangle} = \frac{1}{2} (\mathbb{1} - P_{|a=+1\rangle}),$$

$$= \frac{1}{2} |a = -1\rangle \langle a = -1|,$$  

(7)

where $P_{|A=-1\rangle}$ is the projection on states orthogonal to $|A = -1\rangle$. If the result of the POVM is that corresponding to $E_{A+}$, this means that the initial state was not $|A = -1\rangle$ [35,36]. From the point of view of EPR’s criterion of elements of reality, both $A$ and $a$ must have predefined values +1 or -1; thus any measurement which reveals that $A$ was not -1 also reveals that $A$ was +1. Therefore, following EPR’s point of view, we can label the corresponding output of the POVM as $A = +1$, and likewise for the other three possible outcomes, as in Fig. 2.

However, not only $A$ but also $a$ must have elements of reality. Why then do we obtain only one of them after a POVM? To answer this question, it is useful to keep in mind the fact that any desired POVM with a finite number of elements can be converted into a projective measurement by introducing an auxiliary, independently prepared, quantum system (ancilla) [31, 32,37]. The POVM can then be seen as a projective measurement on the system composed by the original particle and the ancilla. One way of implementing the POVM given by (4)–(7) is by measuring the observable

$$\hat{O} = r_{A+} P_{|A=+1,z=+1\rangle} + r_{A-} P_{|A=+1,z=-1\rangle}$$

$$+ r_{a+} P_{|a=+1,z=+1\rangle} + r_{a-} P_{|a=+1,z=-1\rangle},$$  

(8)

where $P_{|A=+1,z=+1\rangle}$ is the projector onto state $|A = +1\rangle$ of the particle and state $|z = +1\rangle$ of the ancilla, and $r_{A+}$ is the corresponding result. One of the possible ways to measure $\hat{O}$ is by preparing the ancilla in the maximally mixed state $\rho = \frac{1}{2} \mathbb{1}$, then measuring $z$ on the ancilla and then measuring $A$ (if the result of the previous measurement is $z = +1$ or $a$ (if the result is $z = -1$) on the particle. Such a procedure is analogous to the one followed in a standard test of Bell’s inequality with alternative projective measurements. The result of the first measurement acts as a random generator (the two possible outcomes are unpredictable and have the same probability of occurring) which determines the projective measurement that is finally chosen. Therefore, a measurement of $\hat{O}$ is equivalent to a selection between $A$ and $a$ using the result of a projective measurement on the ancilla, followed by a projective measurement of either $A$ or $a$ on the particle. The randomness is provided by a quantum measurement on the ancilla. The result of this measurement selects one experiment or other on the particle. Therefore, we conclude that the described implementation of the POVM (4)–(7) on particle 1 of a two-qubit system in the singlet state is, from the point of view of EPR’s criterion of elements of reality, equivalent to two alternative dichotomic projective measurements $A$ or $a$ preceded by a device to randomly choose between them.

Let us assume that every implementation of a projective measurement is equivalent. Then, the POVM (4)–(7) can be equivalently measured by a single projective measurement of the observable $\hat{O}$ on the particle–ancilla system. In this case, the usual selection-measurement process in each of the wings of a test of Bell’s inequality is replaced with a single measurement on a particle–ancilla system.

### 3. Violating Bell’s inequality with POVMs

The next step is to show that the predictions of quantum mechanics for a singlet state violate the CHSH inequality (2) when each local observer measures a POVM of the type (4)–(7). Let us suppose that the POVM $\alpha$ defined in (4)–(7) is measured on particle 1, and a similar POVM $\beta$, defined as in (4)–(7) just by changing $A$ by $B$ and $a$ by $b$, is measured on particle 2 (Fig. 3). The average $AB$ (and similarly the

![Fig. 2. Modified EPR–Bohm experiment: the measurement on the left particle is a projective measurement, while the measurement on the right particle is a generalized measurement described by the POVM (4)–(7).](image-url)
other three) appearing in the CHSH inequality (2) can be calculated as follows:

\[
\langle AB \rangle_\psi = \left[ P_\psi (E_{A+}, E_{B+}) - P_\psi (E_{A+}, E_{B-}) \right.
\]
\[
- P_\psi (E_{A-}, E_{B+}) + P_\psi (E_{A-}, E_{B-}) \left. \right] 
\]
\[
\times \left[ P_\psi (E_{A+}, E_{B+}) + P_\psi (E_{A+}, E_{B-}) \right.
\]
\[
+ P_\psi (E_{A-}, E_{B+}) \left. \right] \right]^{-1}, \tag{9}
\]

where \( P_\psi (E_{A+}, E_{B+}) \) is the probability of the observer of particle 1 obtaining \( A = +1 \), and the observer of particle 2 obtaining \( B = +1 \). The denominator is the probability of the result of the POVM on particle 1 giving a result belonging to \( \{ E_{A+}, E_{A-} \} \), and the result of the POVM on particle 2 giving a result belonging to \( \{ E_{B+}, E_{B-} \} \). Probabilities for the outcomes obey the Born rule for POVMs:

\[
P_\psi (E_{A+}, E_{B+}) = \langle \psi | E_{A+} \otimes E_{B+} | \psi \rangle. \tag{10}
\]

Therefore, it is easy to see that, for the singlet state (1),

\[
\langle AB \rangle_\psi = - \cos \theta_{AB}, \tag{11}
\]

where \( \theta_{AB} \) is the angle between \( \vec{A} \) and \( \vec{B} \). Choosing \( A = (1, 0, 0), \ a = (0, 0, 1), \ B = (-1, 0, 1) / \sqrt{2}, \) and \( b = (1, 0, 1) / \sqrt{2}, \) we obtain \( \theta_{AB} = 2 \sqrt{2} \), which violates the CHSH inequality (2). Thus, we conclude that the predictions of quantum mechanics for the singlet state violate the CHSH inequality, even if local observers do not have to choose or switch between alternative projective measurements; a suitable POVM implements such a selection-measurement process in a single step.

4. POVMs and loophole-free tests

4.1. Assumption on the ability of local “hidden” variables

To avoid the locality loophole, Bell stressed the importance of experiments “in which the settings are changed during the flight of the particles” [17]. Contrary to what happens in experiments with passive switches [14,21], in the POVM approach one could not reasonably argue that any particle of the pair could guess the “settings”. One might think that particle 1 could know the hidden-variable state of the ancilla associated to particle 2 and that this state could determine the “setting” on particle 2. However, a similar complaint might be applied to any standard test of the CHSH inequality, only changing “hidden-variable state of the ancilla” to “hidden-variable state of the acousto-optical switch” (or any other supposedly “random” method for switching). In any test of Bell’s inequality we have to assume that there are some limits to the ability of the hidden variables of particle 2 to ascertain what is going to be done to particle 1; otherwise the locality loophole could not be avoided by any conceivable setup.

Explicitly, our approach hinges on an additional assumption: (h) the particle on which POVM \( \alpha \) is going to be measured has no ability to ascertain or dictate whether the element of reality, \( A \) or \( a \), is going to be revealed. Such assumption is equivalent to the following usual assumption in the standard approach: (\( h' \)) the local hidden variables have no capability to ascertain or dictate in advance the final setting of the random mechanism.

Supporting the equivalence of assumptions (h) and (\( h' \)) is the following argument: the POVM can be implemented as a single and projective measurement on the particle–ancilla system. This implies an interaction between a measurement apparatus (of which the ancilla is a part) and the particle. From the perspective of the elements of reality, the result of this interaction has two meanings: on one hand, it reveals a pre-existent value; on the other, it entails a choice. Following EPR, the former can be regarded as an element of reality. However, nothing in the EPR criterion authorizes us to suppose that this “choice” is somehow pre-existent neither in the particle nor in the apparatus–ancilla system. The “choice” is not an EPR element of reality. It is the result of an essentially uncontrollable, intrinsically unrepeatable, and basically unpredictable phenomenon which involves interactions between the particle, the apparatus, the ancilla, and the environment in a particular region of space–time. A phenomenon which, to our present knowledge, can be used as a source of randomness [38,39] which offers the ad-
vantage over conventional randomness sources (like electronic noise) of being robust and invulnerable to environmental perturbations. Why then not judge as-sumption \((h)\) to be at least as limiting, if not less, than the usual assumption \((h')\)?

4.2. Practical advantages

At first sight, the standard approach, based on a random selection between two alternative projective measurements, and the approach introduced here, based on single POVM implemented by a single projective measurement on the particle–ancilla system, might be considered to be physically equivalent. However, the equivalence is not complete. The difference could be useful (or at least must be taken into account) in designing loophole-free tests of Bell’s inequality.

Detection efficiencies that are high enough to avoid the detection loophole are usually associated to entangled pairs of massive particles. However, preparing and preserving entanglement between two distant massive particles is a difficult task. Therefore, an interesting question is: which is the minimum distance between the local measurements on the particles necessary to avoid the locality loophole? This distance is \(ct\), where \(c\) is the speed of light in a vacuum. In the standard approach, \(t = t_S + t_M\), where \(t_S\) is the time which elapses since the moment in which the final setting of the selector can be ascertained until the particle enters the measurement device, and \(t_M\) is the time that the projective measurement lasts. In the POVM approach, it is just \(t = t_M\).

Moreover, while \(t_S\) cannot be neglected because it involves a sequence of physical processes (it requires, among other things, connecting a mechanism to generate random results with another which fixes the selector), \(t_M\) is, in principle, comparatively smaller. The behavior of a quantum system subject to a projective measurement is described by von Neumann’s reduction postulate [40]. In this description, measurement is instantaneous. However, real measurements require interactions between the system being measured and the environment, interactions which are not actually instantaneous. However, for our discussion, the important point is that there is no fundamental limit to the minimum time required for a measurement. Therefore, in the POVM approach to test Bell’s inequality, in principle, there would be no limit to the minimum spatial distance between local measurements. In practice, this means that, in the POVM approach, this distance is only restricted by the duration of a single projective measurement.\(^4\)

5. Conclusions

I have proposed a different approach to test the Bell–CHSH inequality. It basically consists in replacing each of the usual pairs of alternative projective measurements preceded by a random mechanism to select between them, by a single fixed POVM. The basic assumption tested by the proposed experiment is the existence of elements of reality as defined by EPR. In Section 2, it has been shown that the POVM \(\alpha\) reveals the value of an EPR element of reality, either \(A\) or \(a\). Since the CHSH inequality is derived from the assumption of the existence of EPR elements of reality, the violation of the CHSH inequality showed in Section 3 can be interpreted as a test of the nonexistence of EPR elements of reality.

However, the argument hinges on an additional assumption: \((h)\) the particle on which the POVM \(\alpha\) is going to be measured has no capability to ascertain or dictate which of the element of reality, \(A\) or \(a\), is going to be revealed. In Section 4, it is argued that such an assumption is equivalent to the usual assumption in the standard approach: \((h')\) the local hidden variables have no capability to ascertain or dictate in advance the final setting of the random mechanism. Any test of Bell’s inequality requires an assumption of this kind. It is stressed that \((h)\) imposes the same or fewer restrictions than \((h')\).

Finally, the advantages of this approach are twofold: (a) it does not require us to assume observers having “free-will”, and (b) it can potentially be used to improve the perspectives of success in the race for a loophole-free test of Bell’s inequality. This is because the new approach eliminates the requirement of a “random” external mechanism before a projective measurement and replaces both the random mechanism

\(^4\) We have assumed that every projective measurement on a composite system can be implemented by a single step process. However, POVMs are sometimes implemented as a two-step process (like, for instance, in the case of ions) and thus no time would be gained.
and the projective measurement itself with a single projective measurement. For some physical systems, at least, this implies a reduction of the spatial length between the local measurements needed to avoid the locality loophole. Such a reduction is of practical interest, since the typical physical systems which allow us to avoid the detection loophole (namely, entangled massive particles) do not allow a significant separation among parts.

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