HOW MANY QUESTIONS DO YOU NEED TO PROVE THAT UNASKED QUESTIONS HAVE NO ANSWERS?

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Suppose a quantum system is prepared in an arbitrary quantum state. How many yes-no questions about that system would you have to consider to prove that such questions have no predefined answers? Peres conjectured that the minimum number was 18, as in the case of the set found in 1995. Asher’s conjecture has recently been proven correct: there are no sets with fewer than 18 questions. This is the end of a long story which began in 1967, when Kochen and Specker found a similar set requiring 117 questions.

Keywords: Kochen–Specker theorem.

1. The Kochen–Specker Theorem

The Kochen–Specker\textsuperscript{1,2} theorem asserts that it is impossible to associate a definite answer (yes or no) to every yes-no question (or test) on a quantum system in such a way that, for every complete set of compatible yes-no questions, one and only one of the answers is yes.

The thrust of this theorem is that any purported physical theory which would attribute a definite answer to each question, and still reproduce the predictions of quantum mechanics, must necessarily be contextual. Namely, if $A$ and $B$ are compatible questions, and $A$ and $C$ are also compatible, but $B$ and $C$ are incompatible, then the answer to $A$ cannot be independent of whether $A$ is tested alone, or together with $B$, or together with $C$.

Some physicists find contextuality to be a reasonable property for hidden-variable theories. For instance, Bell thought that “there is no a priori reason to believe that the results for $A$ should be the same [independently of what other measurements may be made simultaneously].”\textsuperscript{3} However, as Peres and Ron pointed out, contextual hidden-variable theories have a difficulty: “Suppose that we measure $A$ first and only a later time decide whether to measure $B$ or $C$ or none of them. How can the outcome of the measurement $A$ depend on this future decision?”\textsuperscript{4} Moreover, if the physical system under consideration is a composite system, $A$ and $B$ can be chosen to be questions on a different subsystem. Then, the impossibility
of assigning non-contextual definite answers turns out to be the impossibility of assigning local definite answers. In this scenario, how can the outcome of $A$ depend on a space-like separated decision?

Asher was convinced that the KS theorem was fundamental for understanding quantum theory. He lamented, however, that the KS theorem had much less impact on the physics community than Bell’s celebrated theorem “which is actually weaker than the KS result.” This status could be explained by the fact that the KS article was “couched in a rather esoteric mathematical language” and specially because of “its lengthy proof,” which required 117 yes-no questions.

Fortunately, Asher had a talent for explaining complex things in simple language and the enthusiasm for searching for simpler proofs.

The interest of the physics community for the KS theorem was reignited when Asher and David Mermin found that the celebrated proof of Bell’s theorem without inequalities discovered by Greenberger, Horne and Zeilinger was indeed a proof of the KS theorem for a particular quantum state, and that both theorems admit a simple and unified form.

In 1991, Asher wrote: “Over the years, there were many attempts to reduce the number 117, with meagre results.” Indeed, the list of attempts was impressive. However, around that year, Conway and Kochen found a proof requiring only 31 questions for a Hilbert space of dimension 3, while Penrose and Zimba found a proof with 28 questions for dimension 4. Asher found a different proof with 33 questions in dimension 3, and one requiring just 24 questions in dimension 4. For three years, Asher was the holder of the record. However, in 1994, Michael Kernaghan, working on his thesis found that Asher’s proof contained indeed a simpler one requiring just 20 questions. Later Kernaghan and Peres worked together and found a proof in dimension 8 requiring 36 questions.

Asher had included his 24-question proof in his book on quantum theory. In a list of errata distributed among his friends, he added a note saying that the 24-question proof was “superseded” by Kernaghan’s. Indeed, in the 1995 revised paperback edition of his book, he replaced his proof with Kernaghan’s. By then I was working on my PhD thesis, which included a method to generate proofs of the KS theorem in dimension $d$ starting from proofs in dimension $d-1$. The method was usually more efficient as a smaller number of questions had the proof in the lower dimension. This led us to discover that Peres’ (but not Kernaghan’s) proof

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*a* Asher liked simple things. According to Bennett, “Asher’s most important lesson for other scientists is his very personal interpretation of the principle of Occam’s razor (*Pluralitas no est ponenda sine necessitate*): Make it simple, because I can only understand simple things.”

*b* The late Rob Clifton once said: “Kochen–Specker problems can be addictive and I am an addict!”

*c* K. Schütte in 1965, first reported by K. Svozil in 1994. See also Ref. 16.

*d* J. H. Conway and S. Kochen around 1990, first reported in Ref. 9, p. 114. See also Ref. 25.
indeed contained an even simpler proof, one containing just 18 questions. That
occurred on November 22, 1995. We published it. Curiously, the paper which led
us to the discovery, although sometimes cited, remained unpublished until very
recently. Asher included our proof in the list of corrections to the 1995 edition of

2. 18 Questions Are Enough

Years passed and nobody lowered the 18-question record. Then, in 2003, Asher
conjectured that we will hold the record “probably forever.”

How can we prove that a proof cannot be improved? In 2002, I met John Smolin
in a conference in Montreal. He showed me a diagram with 17 “questions” in
dimension 3 which cannot admit definite yes-no answers satisfying the sum rule
of quantum mechanics. The problem was to ascertain whether or not such a set
corresponds to 17 questions on a quantum system described by a Hilbert space
dimension 3. Smolin had arrived at his set by excluding some sets containing
subsets of questions which are impossible by geometrical arguments. The simplest
of these geometrical arguments is the following: if $A$ is compatible with $B$ and $C$, and
$D$ is also compatible with $B$ and $C$, then, in a Hilbert space of dimension 3,
$A$ and $D$ must be the same yes-no question. Therefore, a proof of the KS theorem
cannot contain sets of vectors (representing questions) $A$, $B$, $C$ and $D$, such that
both $A$ and $D$ are orthogonal to (i.e. compatible with) both $B$ and $C$.

Unfortunately, Smolin’s set was not a proof of the KS theorem because it con-
tained a subset of ten questions which was impossible due to geometrical reasons
(the details can be found in Ref. 37). The interesting thing was that Smolin’s com-
puter search had been exhaustive: the only set of 17 “questions” in dimension 3
not containing impossible subsets was (modulo isomorphisms) the one Smolin had
found. Therefore, this proved that there was no proof with fewer than 18 questions
in a Hilbert space of dimension 3.

What about dimension 4? In 2005, Pavičić et al. presented a set of algorithms for
exhaustively generating all possible proofs of the KS theorem in any dimension. To
demonstrate their usefulness, they implemented them and explored the possible
proofs of the KS theorem in dimension 3 (up to 30 questions) and dimension 4 (up to
24 questions). The complexity of this computer search grows exponentially with the
number of questions. They found no proof in dimension 3, and that the 18-question
proof is the smallest one in dimension 4. Therefore, this proved that there was
no proof with fewer than 18 questions in a Hilbert space of dimensions 3 or 4.

What about higher dimensions? The answer is that there is no proof of the KS
theorem with fewer than 18 questions in any dimension. The proof is based on two
results. On the one hand, the observation that each question appearing in a proof of
the KS theorem, in any dimension, must belong to a complete set of questions also
appearing in that proof and, in addition, must be compatible with another question
not belonging to that complete set. On the other hand, the observation that there
are simple sets of questions that are impossible due to geometrical reasons and which can be generated in a recursive manner from similar ones in lower dimensional spaces. In addition, it is useful to note that there is a one-to-one correspondence between sets of questions, some of them mutually compatible, and graphs (sets of dots connected by segments). Nowadays, there are computer programs for isomorphism-free exhaustive generation of graphs,\cite{39,40} where the previous observations can be efficiently implemented. An exhaustive search shows that there is no proof with fewer than 18 questions in any dimension.\cite{37}

3. Which 18 Questions?

The proof of the KS theorem with 18 yes-no questions, each represented by a one-dimensional projection operator in a Hilbert space of dimension 4,\cite{31,32,41,42} is reproduced in Table 1.

Table 1 contains 18 vectors combined in 9 columns. Each vector appears twice. Each vector represents the projection operator onto the corresponding normalized vector. For instance, 001\bar{1} represents the projector onto the vector $\frac{1}{\sqrt{2}}(0, 0, 1, -1)$. Each column contains four mutually orthogonal vectors, so that the corresponding projectors sum the identity. Therefore, in a non-contextual hidden variable theory, each column must have the answer “yes” assigned to one and only one vector. But it is easily seen that such an assignation is impossible, since each vector appears twice; therefore, the total number of “yes” answers must be an even number.

Which questions are these? The quantum system on which this set of questions allows us to prove the KS theorem must have, at least, four degrees of freedom. It can be, for instance, the spin state of a single spin-$3/2$ particle, or the spin state of two spin-$1/2$ particles (i.e. two qubits). In this second scenario, it can be easily realized that the 18 vectors in Table 1 are eigenvectors of some products of the usual representation of the Pauli matrices $\sigma_z$ and $\sigma_x$ for the spin state of spin-$1/2$ particles. This allows us to rewrite Table 1 as Table 2, which contains the meaning of the questions appearing in the proof.

<table>
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<th>Table 1. The 18-question proof of the KS theorem.</th>
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<tr>
<td>1000 1111 1111 1000 1001 1001 1111 1111 1001</td>
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<tr>
<td>0100 1111 1111 0010 0100 1111 1100 0101 0110</td>
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<tr>
<td>0011 1100 1010 0101 0010 1111 0011 1010 1111</td>
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<tr>
<td>001\bar{1} 001\bar{1} 001\bar{1} 001\bar{1} 001\bar{1} 001\bar{1} 001\bar{1} 001\bar{1} 001\bar{1}</td>
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<th>Table 2. The 18-vector proof of the KS theorem seen for a system of two qubits.</th>
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<td>$zz$ $zx$ $xz$ $zx$ $zz$ $zx$ $xz$ $zx$ $zz$ $zx$ $xz$ $zx$ $zz$ $zx$ $xz$ $zx$ $zz$</td>
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The notation in Table 2 is the following: $z\bar{x}$ represents the yes-no question “is the spin component of first particle positive in the $z$-direction and the spin component of second particle negative in the $x$-direction?,” and $\bar{z}xz$ denotes the yes-no question “are the products $zx := \sigma_1^z \otimes \sigma_2^x$ and $xz := \sigma_1^x \otimes \sigma_2^z$ negative and positive respectively?” and so on. The first is an example of a product yes-no question, since it can be answered after separate tests on the first and second particles. The latter is an example of an entangled yes-no question, since it cannot be answered after separate tests on both particles. Therefore, in Table 2, there are two types of yes-no questions and, consequently, three types of maximal tests: those involving product yes-no questions only, such as those in columns 1 to 4; those involving both product and entangled yes-no questions, such as those in columns 4 to 8; and those involving entangled yes-no questions only, such as the one in the ninth column. Taking into account this hierarchy of experiments, the relevant elements of the proof of the KS theorem for two qubits can be illustrated as in Fig. 1.

Beyond its graphical beauty, the interest in Fig. 1 arises from the fact that it encapsulates the hierarchy of tests that lies behind a gedanken experiment which challenged the old idea that the KS theorem could not be tested in a laboratory.\textsuperscript{43}

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig1.png}
\caption{Graphical representation of the 18-question proof of the KS theorem seen for a system of two qubits. Each segment represents a yes-no question. Segments meeting at the same point are mutually compatible yes-no questions, and therefore each point represents a maximal test. Each of the four points delimiting a square in the centre is a test containing only product yes-no questions. Each of the two points above and the two points below is a test containing both product and entangled yes-no questions. The point in the centre is a test containing only entangled yes-no questions.}
\end{figure}
Such a gedanken experiment was refined in Ref. 44 and finally performed in Ref. 45. The connection between the 18-question proof and the experiment proposed in Ref. 43 is explained in Ref. 42.

What do we learn from the KS theorem? Asher accurately summed up the lesson in his immortal words: we have learnt that “unperformed experiments have no results.”

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References