Generalized Ardehali-Bell inequalities for graph states

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We derive Bell inequalities for graph states by generalizing the approach proposed by Ardehali [Phys. Rev. A 46, 5375 (1992)] for Greenberger-Horne-Zeilinger (GHZ) states. Using this method, we demonstrate that Bell inequalities with nonstabilizer observables are often superior to the optimal GHZ-Mermin-type (or stabilizer-type) Bell inequalities. 

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I. INTRODUCTION

Bell inequalities are constraints imposed by local hidden variable (LHV) models on the correlations of distant experiments. The fact that quantum mechanics predicts a violation of these relations makes them useful for demonstrating the impossibility of LHV models. In addition, Bell inequalities are useful as entanglement witnesses of these relations makes them useful for demonstrating the security of some quantum key distribution systems. The fact that quantum mechanics predicts a violation has been achieved only for some specific cases of the subject of intensive research, and a complete classification of Bell inequalities is the one maximally violated by this state, be

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Moreover, finding Bell inequalities with a high amount of violation for a given state allows us to investigate the interplay between the violation of local realism and decoherence. Greenberger, Horne, and Zeilinger (GHZ) showed that multipartite states which are simultaneous eigenstates of several local observables (henceforth called GHZ states) can lead to striking contradictions with local realism if we consider the perfect correlations between these local observables [8]. The idea to construct Bell inequalities from such perfect correlations has been extended by Mermin and others into several directions [9–17]. The extensions include GHZ states with more particles, cluster states, and graph states—which generalize GHZ states and are of great importance for many applications in quantum information [18]. Remarkably, as early as in 1992, Ardehali showed that for special examples of GHZ states, other Bell inequalities exist which lead to a higher violation of local realism compared to the GHZ-Mermin-type (or stabilizer-type) inequalities [19]. In this paper we generalize Ardehali’s method to arbitrary graph states. This allows us to derive Bell inequalities with a high amount of violation for a variety of different states. We find that the inequalities are often superior to the optimal Bell inequalities of the GHZ-Mermin-type inequalities, suggesting that the curious property discovered by Ardehali is quite generic. Interestingly, although the Ardehali approach was originally designed to derive Bell inequalities which use not only perfect correlations, our extended method also allows us to derive some Bell inequalities of the stabilizer type for tree graph states (i.e., graph states associated to graphs which do not contain any closed loop).

This paper is organized as follows: In Sec. II we give a short definition of graph states. In Sec. III we reformulate Ardehali’s method in the language of stabilizing operators and graph states. Then, in Sec. IV we apply the method to obtain Bell inequalities for five- and six-qubit graph states, showing that the Ardehali approach delivers higher violations of local realism than GHZ-Mermin-type inequalities. In Sec. V we further extend the method and derive some general Bell inequalities for tree graph states. Finally, in Sec. VI we sum up the results.

II. GRAPH STATES

Graph states are a family of multiqudit states that play a crucial role in many applications of quantum information theory, such as quantum error correcting codes, measurement-based quantum computation, and quantum simulation [18,20]. Consequently, a significant experimental effort is devoted to the creation and investigation of graph states [21–25]. Graph states are defined as follows:

Let G be a graph, i.e., a set of N vertices corresponding to qubits and edges connecting them. Some examples are shown in Figs. 1–3. For each vertex i, the neighborhood N(i) denotes the vertices which are connected with i. Then, we can associate a stabilizing operator g_i to each vertex i by

\[ g_i := X^{(i)} \otimes_j N(i) Z^{(j)}. \] (1)

Here and in the following, X^{(i)}, Y^{(i)}, Z^{(i)} denote the Pauli matrices \( \sigma_x, \sigma_y, \sigma_z \), acting on the ith qubit. The index (i) may be omitted where there is no risk of confusion. The graph state \( |G\rangle \) associated with the graph G is the unique N-qubit state fulfilling...
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FIG. 1. (Color online) The star graph for (a) four and (b) six
vertices. The corresponding graph state is, up to local unitary trans-
formations, the GHZ state. The Ardehali inequality for the six-qubit
GHZ state can be derived from the Mermin inequality for the five-
qubit GHZ state, when a sixth qubit is added. See Sec. III for
details.

\[ g_i |G\rangle = |G\rangle, \quad \text{for } i = 1, \ldots, n. \] (2)

Physically, the stabilizing operators describe the perfect cor-
relations in the state |G\rangle, since \( \langle g_i \rangle = \langle X^{(i)} \otimes \otimes_{j \neq i} Z^{(j)} \rangle = 1 \). In
the GHZ-Mermin-type (or stabilizer-type) Bell inequalities, these
perfect correlations are used to derive contradictions to
local realism [8–14].

For our later discussion, it is important to note that differ-
ent graphs may lead to graph states which differ only by a
local unitary transformation; that is, their entanglement prop-
erties are the same. The main graph transformation, which
leaves the entanglement properties invariant, is the so-called
local complementation [18].

Local complementation acts on a graph as follows: one
picks out a vertex \( i \) and inverts the neighborhood \( \mathcal{N}(i) \) of \( i \); that is, vertices in the neighborhood which were connected
become disconnected and vice versa. It has been shown that
local complementation acts on the graph state as a local uni-
tary transformation of the Clifford type, and therefore leaves
the nonlocal properties invariant. More specifically, it in-
duces the map \( Y^{(i)} \mapsto Z^{(i)}, Z^{(i)} \mapsto Y^{(i)} \) on the qubit \( i \), and the
map \( X^{(j)} \mapsto Y^{(j)}, Y^{(j)} \mapsto X^{(j)} \) on the qubits \( j \in \mathcal{N}(i) \) of the
neighborhood [18]. On the level of the generators, it maps
the generators \( g_j^{old} \) with \( j \in \mathcal{N}(i) \) to \( g_j^{new} g_i^{new} \). We will see
later an explicit example of this transformation. Finally, it
should be noted that not all local unitary transformations

\begin{align*}
N & = 2 \left( \frac{1}{2} \right)^{N-2} |00\ldots 0\rangle + |11\ldots 1\rangle, \\
& = |GHZ\rangle.
\end{align*}

(3)

which is an eigenstate with eigenvalue one of the stabilizing
operators,

\begin{align*}
g_1 & = X^{(1)} X^{(2)} \ldots X^{(N)}, \\
g_2 & = Z^{(1)} Z^{(2)} Z^{(3)} \ldots Z^{(N)}, \\
g_3 & = Z^{(1)} Z^{(2)} Z^{(3)} \ldots Z^{(N)}, \\
g_N & = Z^{(1)} Z^{(2)} Z^{(3)} \ldots Z^{(N)}.
\end{align*}

These are, up to a local change of the basis, stabilizing op-
erators like those in Eq. (1). The GHZ state corresponds to a
star graph such as those shown in Fig. 1.

A Bell inequality for the GHZ state is the Mermin in-
equality [9], which has the Bell operator

\[ B_N = g_1 \prod_{k=2}^{N} (1 + g_k). \] (5)

According to quantum mechanics, the expectation value of
\( B_N \) for the GHZ state is \( \langle B_N \rangle = 2^{N-1} \), while for local realistic
models \( \langle B_N \rangle_{LHV} \approx C_N \), with \( C_N = 2^{(N-1)/2} \) (C_N = 2^{N/2} \) for \( N \) odd
(even). The amount of violation is then \( V_N = \langle B_N \rangle / C_N = 2^{(N-1)/2} \) \( / / 2^{(N-2)/2} \) for \( N \) odd (even).

We can rewrite the Bell operator (5) as

\[ g_i |G\rangle = |G\rangle, \quad \text{for } i = 1, \ldots, n. \] (2)
Finally, it should be noted that in some cases we may add several qubits consecutively. By adding several qubits one can derive a similar bound as in Eq. (7), and one can then directly obtain a Bell inequality with a violation of \( V_N = (\sqrt{2})^N V_{N-k} \).

IV. APPLICATION TO BELL INEQUALITIES FOR THE FOUR-QUBIT CLUSTER STATE

The method presented in Sec. III does not allow us to obtain new Bell inequalities for four-qubit graph states: The only three-qubit graph state is the GHZ state [18], with the Mermin inequality as the relevant Bell inequality. As can be easily seen, the only way to derive an Ardehali-type Bell inequality for four-qubit states from that result is the Ardehali inequality for the four-qubit GHZ state, which is already known [19].

However, we can apply the method to Bell inequalities for the four-qubit cluster state and obtain Ardehali-type Bell inequalities for different five-qubit graph states.

A. Five-qubit states

Let us start by applying the method to the Bell inequalities for the four-qubit (linear) cluster state derived by Scarani et al. [10,12–14]. The graph of this state is shown in Fig. 2. One of the suitable inequalities is given [14] by

\[ B_1^{(LC4)} = (1 + g_1)g_2(1 + g_3)g_4 \]  

\[ = ZXIX - ZYYX + YY1X + YXXY. \]  

Here, we omit the indices on the Pauli operators for simplicity. This Bell inequality leads to a violation \( V_4 = 2 \), since for the cluster state \( B_4^{(LC3)} = 4 \), and the bound for LHV models is 2. Since \( g_4 \) is a factor of the Bell operator, we can add a qubit at the end [see Fig. 2(a)] and obtain the Bell operator

\[ B_1^{(LC5)} = (1 + g_1)g_2(1 + g_3)g_4(1 + g_5) \]

\[ = [(ZXIX - ZYYX + YY1X + YXXY)(A + B) \]

\[ + (ZX1Y + ZYYX + YY1Y - YXXX)(A - B)]/\sqrt{2}, \]  

with \( A = (Z+Y)/\sqrt{2} \) and \( B = (Z-Y)/\sqrt{2} \) [28]. This Bell operator requires a measurement of 16 correlation terms. It has a maximum value of 2\( \sqrt{2} \) for LHV models, leading to a violation \( V_5 = 2\sqrt{2} \approx 2.82 \) for the five-qubit linear cluster state. Remarkably, if only stabilizing operators are considered, it can be proven that the maximal achievable violation is only \( 5/2 = 2.5 \) [14]. Therefore, for the five-qubit linear cluster state, the violation of local realism can be increased by considering Ardehali-type inequalities, as for the GHZ state with a even number of qubits.

Since \( g_2 \) is also a factor of \( B_1^{(LC4)} \), we can also connect the fifth qubit to the second qubit [see Fig. 2(b)]. This leads to a Bell inequality for the five-qubit \( Y \) state.

\[ B_1^{(LC5)} \]
\[ \mathcal{B}^{(V)} = [(ZYX - ZYXY + YYIX + YXXY)(A + B) + (ZYX + ZXY - XYIX + YXY)(A - B)]/\sqrt{2}, \]

(12)

with \( A = (Z + Y)/\sqrt{2} \) and \( B = (Z - Y)/\sqrt{2} \) [28]. Therefore, the five-qubit \( Y \) state also has a violation of \( V = 2.7\sqrt{2} \approx 2.82 \). If only stabilizing operators are considered, the maximal achievable violation is only \( 7/3 \approx 2.33 \) [14], proving again the superiority of the method presented here.

Finally, a further inequality is given by the Bell operator \( \mathcal{B}_{L}^{(4)} = (1 + g_1)(g_2 + g_4) \) [14]. Here, \( (g_1 + g_4) \) is a factor of \( \mathcal{B}_{L}^{(4)} \), hence we can add a qubit connected to qubits 3 and 4 at the same time, as shown in Fig. 2(c). The resulting five-qubit state also has a violation of \( 2.82 \). The Bell operator is given by \( \mathcal{B}_{L}^{(5)} = (1 + g_1)(g_2 + g_4)(1 + g_3) \), where we have to introduce the measurements \( A = (Z + Y)/\sqrt{2} \) and \( B = (Z - Y)/\sqrt{2} \) on the fifth qubit.

This state, however, was discussed before. Local complementation on the vertex 4 requires us to invert its neighborhood, consisting of vertices 3 and 5, which become disconnected. Therefore, the state in Fig. 2(c) is equivalent to the five-qubit linear cluster state in Fig. 2(a). If we make the local complementation, we obtain \( \mathcal{B}_{L}^{(5)} = (1 + g_1)(g_2 + g_4)(1 + g_3) \), with the measurements \( A = (Z + Y)/\sqrt{2} \) and \( B = (Z - Y)/\sqrt{2} \) on the fifth qubit, as a Bell operator for the linear cluster state.

B. Six-qubit states

Let us now demonstrate how the method presented here can be used to derive Bell inequalities for six-qubit states directly from the four-qubit inequalities.

First, as already mentioned, one can add two qubits consecutively to the cluster state and obtain the six-qubit \( Y \) state [see Fig. 2(d)]. Taking \( \mathcal{B}_{L}^{(6)} \) as above, considering \( \mathcal{B}_{L}^{(6)} = (1 + g_1)(g_2 + g_4)(1 + g_3)(1 + g_5) \) and replacing \( A = (Z + Y)/\sqrt{2} \) and \( B = (Z - Y)/\sqrt{2} \) on the fifth and sixth qubit, one finds that the depicted six-qubit state has a violation of \( V = 6 \). Note that the state in Fig. 2(d) is of special interest in quantum information science, since it allows a demonstration of anyonic statistics in the Kitaev model [20,24,25].

Another interesting example is shown in Fig. 2(e). First, by a suitable sequence of local complementations and relabeling of the qubits (first one makes a local complementation on the second qubit, then on the third, and finally again on the second), one can transform the graph of a four-qubit cluster state in the form of a “box,” as shown here. Afterwards, the Bell inequality in Eq. (10b) reads

\[ \mathcal{B}_{Box}^{(4)} = g_1(1 + g_2)(1 + g_3). \]

(13)

We can add a fifth qubit as in Fig. 2(e) and achieve a violation of \( V = 2.82 \). Clearly, if we add a sixth qubit as shown in Fig. 2(e), but do not change the Bell operator, the violation is unchanged. The point is that, if only stabilizing operators are considered, the graph state in Fig. 2(e) leads only to a violation of \( 2.5 \) [14], showing that the method presented here can also bring improvements for six-qubit states.

V. THE GENERALIZED METHOD

Getting back to the derivation of the first inequality in Eqs. (6b) and (7), we note that we can extend said derivation in two directions. First, in Eq. (6b) it is not necessary that the Bell operator \( \mathcal{B}_{L}^{(N-1)} \) originate from \( \mathcal{B}_{L}^{(N-1)} \) via some transformation. In principle, we can choose \( \mathcal{B}_{L}^{(N-1)} \) independently of \( \mathcal{B}_{L}^{(N-1)} \). Second, in Eqs. (7), to obtain a bound for LHV models, we used an argument similar to the one in the Clauser-Horne-Shimony-Holt inequality [29]. Here, we can use more general inequalities, such as the Mermin-Ardehali-Belinskii-Klyshko inequalities [5].

A. Linear cluster states

Let us consider the situation in Fig. 3(a), where two connected qubits are added to a four-qubit cluster state. For the four-qubit cluster state, we consider two possible Bell operators: first \( \mathcal{B}_{L}^{(4)} \) from Eq. (10b), and further \( \mathcal{B}_{L}^{(4)} = (1 + g_1)(g_2 + g_5) \), which also has a bound for LHV models of \( 2 \) [10,12–14]. For the graph state in Fig. 3(a), we consider the six-qubit operator

\[ \mathcal{B}_{L}^{(6a)} = \mathcal{B}_{L}^{(4)}(1 + g_5) + \mathcal{B}_{L}^{(4)}(g_5 + g_6) \]

(14a)

\[ \mathcal{B}_{L}^{(6a)} = \mathcal{B}_{L}^{(4)}(1 + g_5)g_6 + \mathcal{B}_{L}^{(4)}g_5(1 + g_6), \]

(14b)

where \( \mathcal{B}_{L}^{(4)} \) denotes the restriction of \( \mathcal{B}_{L}^{(4)} \) on the first four qubits, and \( \mathcal{B}_{L}^{(4)} \) is the Bell operator obtained from \( \mathcal{B}_{L}^{(4)} \) by exchanging \( Z \to 1 \) on the fourth qubit.

The point of the ansatz of Eq. (14a) is that the inequality can be considered as a three-body Mermin inequality, where \( \mathcal{B}_{L}^{(4)} \) and \( \mathcal{B}_{L}^{(4)} \) take the role of the observables on the first party. Since the maximum values for LHV models for \( \mathcal{B}_{L}^{(4)} \) and \( \mathcal{B}_{L}^{(4)} \) are 2, the bound for the total Bell operator \( \mathcal{B}_{L}^{(6a)} \) is 4, and the graph state in Fig. 3(a) violates local realism by a factor of 4.

After a local complementation on the fifth qubit, the state in Fig. 3(a) is equivalent to the six-qubit linear cluster state in Fig. 3(b). The transformed Bell operator reads

\[ \mathcal{B}_{L}^{(6b)} = (1 + g_1)(g_2 + g_5)(1 + g_6), \]

(15)

and has only stabilizing operators [14]. In fact, it can be seen as a product of two three-qubit Mermin operators.

Interestingly, this method may be iterated as follows. If we consider the seven-qubit linear cluster state, we can write down two Bell inequalities leading to a violation of 4: First, we use the six-qubit Bell operator from Eq. (15), i.e., \( \mathcal{B}_{L}^{(7)} = (1 + g_1)(g_2 + g_5)(1 + g_6) \), then we can use its product with the generator \( g_7 \), i.e., \( \mathcal{B}_{L}^{(7)} = (1 + g_1)(g_2 + g_5)(1 + g_6)g_7 \). Taking both Bell operators, we can use again the trick used in Fig. 3(a) to obtain a Bell inequality for a nine-qubit graph state. After local complementation, this leads to a Bell inequality for the nine-qubit linear cluster state, which is the product of three three-qubit Mermin inequalities.

Finally, it should be noted that it is not trivial that the product of two three-qubit Mermin operators is a Bell opera-
tor in which the total violation is the product of the two violations, since the three-qubit graphs are connected and not independent. To give an example, if we consider the Bell operator in Eq. (15) and connect the first and sixth qubit (in order to form a six-qubit ring cluster state), it will not lead to a violation by a factor of 4, but to a violation by only a factor of 2. In general, one can prove (along the lines of the proof of Lemma 3 in Ref. [11]) that, if \(|B_1|_{LHV} \leq C_1\) and \(|B_2|_{LHV} \leq C_2\) are two Mermin-GHZ-type inequalities on two graphs \(G_1\) and \(G_2\) and these graphs become connected by a single edge, then the Bell operators can be multiplied and the threshold for LHV models is the product of \(C_1\) and \(C_2\).

B. Six-qubit Y state

Another way to obtain a generalized Ardehali inequality is shown in Fig. 3(c). Here, we take the Bell operators \(B_2^{(LC4)} = (1 + g_1) g_2 (g_3 + g_4)\) and \(B_4^{(LC4)} = (1 + g_1) g_2 (1 + g_3 g_4)\), which are both bounded by 2 for LHV models, and consider

\[
B^{(3c)} = B_2^{(LC4)} (1 + g_5 g_6) + B_4^{(LC4)} (g_5 + g_6) \quad (16a)
\]

\[
= [B_2^{(LC4)}]_{1,4} (ZZ - XX) + [B_4^{(LC4)}]_{1,4} \times (XZ + ZX), \quad (16b)
\]

where \([B_4^{(LC4)}]_{1,4} = ZX|Z + YY|Z = ZYYX + YYYX\) is a transformed version of \(B_4^{(LC4)}\). With the same argumentation as before, we have for LHV models \(B^{(3c)} \leq 4\), so the state in Fig. 3(c) has a violation by a factor of 4.

This state can be transformed to the six-qubit Y state of Fig. 2(d) by local complementation on the fourth qubit. Transforming the Bell operator \(B^{(3c)}\) accordingly, one arrives at the Bell operator

\[
B^{(Y6)} = (1 + g_1) g_2 (1 + g_3) g_4 (1 + g_5) (1 + g_6), \quad (17)
\]

which contains only stabilizing operators [14]. It leads to a violation of 4, even if the replacement \(A = (Z + Y)/\sqrt{2}\) and \(B = (Z - Y)/\sqrt{2}\) is not done. Again, one can iterate this procedure in order to obtain Bell inequalities with a high amount of violation for tree graph states (i.e., graph states associated to graphs which do not contain any closed loop).

C. Further extensions

Let us briefly mention further extensions and applications of the presented method. First, we can easily derive new Bell inequalities with a high amount of violation for the case in which three or more connected qubits are added [see Fig. 3(d)]. Then, however, one often does have to make the replacement \(A = (Z + Y)/\sqrt{2}\) and \(B = (Z - Y)/\sqrt{2}\) in order to obtain a Bell inequality with a high amount of violation.

Further, the presented methods can be used if several graphs become connected. As shown in Fig. 3(e) one may take three three-qubit graphs, and connect them via one additional qubit. Proper Bell operators for each of the three qubit graphs are \(B_1 = (1 + g_1) g_2 (1 + g_3)\), etc., and a Bell operator for the three disconnected graphs is \(B = B_1 B_2 B_3\), leading to a total violation by a factor of 8. For the ten-qubit graph shown in Fig. 3(e), we can directly write an Ardehali-like Bell inequality, with a violation of \(8 \sqrt{2}\). Similarly, one can build two- or three-dimensional structures out of smaller graphs, and obtain for them Bell inequalities with a high amount of violation. Such larger structures are of great interest, since one-dimensional structures (such as the linear cluster state) are typically not universal resources for measurement-based quantum computation [30,31]. Further results on this problem will be given elsewhere.

VI. CONCLUSION

In conclusion, we have introduced a general method for deriving Ardehali-type Bell inequalities for graph states. We have applied the method to a variety of different graphs and showed that the obtained inequalities often lead to a higher violation of local realism than stabilizer-type Bell inequalities. We have generalized the method and also obtained Bell inequalities with a high amount of violation for tree graph states. Deriving Bell inequalities with a high amount of violation for graph states associated to general two-dimensional lattices remains an interesting and open problem for further study.

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[28] Since we are working now in the original graph state formalism, we have to make the replacement $X \mapsto Y; Y \mapsto -X$ to obtain $\tilde{B}$ from $B$.