State-independent quantum contextuality for continuous variables

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Recent experiments have shown that nature violates noncontextual inequalities regardless of the state of the physical system. So far, all these inequalities involve measurements of dichotomic observables. We show that state-independent quantum contextuality can also be observed in the correlations between measurements of observables with genuinely continuous spectra, highlighting the universal character of the effect.

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I. INTRODUCTION

Recent experiments [1–5], following the proposal in [6], have shown that nature cannot be described by noncontextual theories and that this impossibility can be detected by a state-independent violation of an inequality. The motivation behind these experiments comes from the observation made by Kochen and Specker [7,8] and Bell [9] that contextuality is a necessary property in any attempt to complete quantum mechanics (QM) with additional variables. In a similar way, nonlocality is a necessary property in any attempt to complete QM [10]. However, while nonlocality is only needed to explain the quantum predictions when the physical system is prepared in an entangled state, contextuality is needed to explain the quantum predictions regardless of which state the system is in. The recent developments reported in [1–6,11] are helping to overcome the obstacles for the experimental study of quantum contextuality that have been pointed out in the literature [12–16] (see [17] for a detailed discussion).

So far, all state-independent violations of noncontextual inequalities [1,3,5,6,11,18,19] invoke dichotomic observables. Moreover, the original proofs of the impossibility of noncontextual alternatives to QM are only valid for observables with discrete spectra [8,9]. The case of continuous variables like position or momentum is of fundamental importance since “all measurements of quantum mechanical systems could be made to reduce eventually to position and time measurements” [20] or, according to Bell, “in physics the only observations we must consider are position observations” [21]. There is an extensive literature on Bell inequalities for local hidden-variable theories with continuous variables [22–24]. Additional motivations behind these researches are (a) to extend the range of quantum states violating the inequalities and achieve the maximal violation [23] and (b) to avoid the need of dichotomic binning of the results to get a violation [24]. On the other hand, the extension of quantum information to continuous variables has attracted great interest, since it has important technological implications [25–28].

Therefore, a fundamental question is whether there is a state-independent violation of a noncontextual inequality using only continuous variables. In this article we derive a simple noncontextual inequality for continuous variables such that, according to QM, there is a universal set of observables for which (a) any state maximally violates the inequality and (b) the violation does not require any binning of the results.

II. INEQUALITY

Consider 18 observables, $A', A'', B', B'', C', C''$, $a', a'', b', b'', c', c'', a', a'', b', b'', c'$, and $\gamma''$, which take any possible value between $-1$ and $1$:

$$-1 \leq A' \leq 1, \ldots, \quad (1a)$$

$$-1 \leq \gamma'' \leq 1. \quad (1b)$$

In addition, these values are assumed to satisfy the following restrictions:

$$|A' + iA''| = 1, \ldots, \quad (2a)$$

$$|\gamma' + i\gamma''| = 1, \quad (2b)$$

where $i$ is the imaginary constant and $|x|$ denotes the modulus of the complex number $x$. For convenience, hereafter we use the following notation:

$$A = A' + iA'', \ldots, \quad (3a)$$

$$\gamma = \gamma' + i\gamma'', \quad (3b)$$

and we consider mean values like $\langle ABC \rangle = \langle A' + iA'' \rangle \langle B' + iB'' \rangle \langle C' + iC'' \rangle$, where $A', A'', B', B'', C'$, and $C''$ are mutually compatible observables. To experimentally obtain $\langle ABC \rangle$, one has to sequentially measure the six observables on the same individual system and then compute the complex number $(A' + iA'') (B' + iB'') (C' + iC'')$. Then, one must repeat the experiment many times on identically prepared copies and take the average of the real part and the average of the imaginary part.

Lemma. Any theory in which all these 18 observables have predetermined noncontextual outcomes (i.e., independent of which compatible observables are jointly measured) must satisfy the following inequality:

$$|\langle ABC \rangle + \langle abc \rangle + \langle a\beta\gamma \rangle + \langle Aa\alpha \rangle + \langle Bb\beta \rangle - \langle Cc\gamma \rangle| \leq 3\sqrt{3}, \quad (4)$$

where the observables inside each mean value are mutually compatible.
Proof. To obtain the upper bound of inequality (4), let us first find the maximum possible value of

\[
|S| = |ABC + abc + αβγ + Aαα + Bββ - Ccγ|,
\]

\[
= |A(BC + aα) + b(ac + Bβ) + γ(αβ - Cc)|,
\]

where \(A, B, C, a, b, c, α, β,\) and \(γ\) are nine arbitrary complex numbers of modulus 1. Then,

\[
|S| \leq |BC + aα| + |ac + Bβ| + |αβ - Cc|,
\]

since \(|A| = |b| = |γ| = 1.\)

To find an upper bound for the right-hand side of (6), we introduce the phases \(ϕ_1\) and \(ϕ_2\), defined as

\[
\frac{BC}{aα} = e^{iϕ_1},
\]

\[
\frac{ac}{aβ} = e^{iϕ_2},
\]

\[
\frac{Cc}{αβ} = e^{i(ϕ_1 + ϕ_2)}.
\]

From (7), it follows that

\[
|BC + aα|^2 = 4\cos^2\frac{ϕ_1}{2},
\]

\[
|ac + Bβ|^2 = 4\cos^2\frac{ϕ_2}{2},
\]

\[
|αβ - Cc|^2 = 4\sin^2\left[\frac{1}{2}(ϕ_1 + ϕ_2)\right].
\]

From Eqs. (8) we see that finding the maximum of the right-hand side of (6) is tantamount to finding the maximum of

\[
2\left|\cos\frac{ϕ_1}{2}\right| + \left|\cos\frac{ϕ_2}{2}\right| + \left|\sin\frac{1}{2}(ϕ_1 + ϕ_2)\right|.
\]

It can be easily seen that this maximum is \(3\sqrt{3} \approx 5.19\) (for instance, it occurs when \(ϕ_1 = ϕ_2 = π/3\)). Therefore,

\[
|S| \leq 3\sqrt{3}.
\]

Finally, if one repeats the experiment many times on identically prepared copies of the system, then one can use that

\[
|\langle S \rangle| \leq |\langle S \rangle|
\]

and obtain inequality (4). \(\Box\)

The upper bound of (4) can be reached, for instance, for

\[
A' = C'' = b' = c'' = γ' = \frac{\sqrt{3}}{2},
\]

\[
A'' = -C' = b'' = -c' = -γ'' = -\frac{1}{2},
\]

\[
b' = a' = a'' = β'' = 1,
\]

\[
b'' = a'' = a'' = β'' = 0.
\]

III. QUANTUM VIOLATION

Let us consider a quantum-mechanical system consisting of a particle moving in a two-dimensional space. This system has continuous position and momentum observables, \(x = (x_1, x_2)\)

and \(p = (p_1, p_2)\), that comply with the standard canonical commutation relations,

\[
[x_1, x_2] = 0,
\]

\[
[p_1, p_2] = 0,
\]

\[
[x_i, p_j] = i\hbar δ_{ij}.
\]

Now consider the 18 observables described in QM by the following Hermitian operators:

\[
A' = \cos\left(\frac{p_0}{\hbar} x_1\right), \quad A'' = \sin\left(\frac{p_0}{\hbar} x_1\right),
\]

\[
B' = \cos\left(\frac{π}{p_0} p_2\right), \quad B'' = \sin\left(\frac{π}{p_0} p_2\right),
\]

\[
C' = \cos\left(\frac{p_0}{\hbar} x_1 + \frac{π}{p_0} p_2\right), \quad C'' = -\sin\left(\frac{p_0}{\hbar} x_1 + \frac{π}{p_0} p_2\right),
\]

\[
a' = \cos\left(\frac{p_0}{\hbar} x_2\right), \quad a'' = -\sin\left(\frac{p_0}{\hbar} x_2\right),
\]

\[
b' = \cos\left(\frac{π}{p_0} p_1\right), \quad b'' = \sin\left(\frac{π}{p_0} p_1\right),
\]

\[
c' = \cos\left(\frac{p_0}{\hbar} x_2 - \frac{π}{p_0} p_1\right), \quad c'' = \sin\left(\frac{p_0}{\hbar} x_2 - \frac{π}{p_0} p_1\right),
\]

\[
α' = \cos\left(\frac{p_0}{\hbar} (x_2 - x_1)\right), \quad α'' = \sin\left(\frac{p_0}{\hbar} (x_2 - x_1)\right),
\]

\[
β' = \cos\left(\frac{π}{p_0} (p_1 + p_2)\right), \quad β'' = -\sin\left(\frac{π}{p_0} (p_1 + p_2)\right),
\]

\[
γ' = \cos\left(\frac{p_0}{\hbar} (x_1 - x_2) + \frac{π}{p_0} (p_1 + p_2)\right),
\]

\[
γ'' = \sin\left(\frac{p_0}{\hbar} (x_1 - x_2) + \frac{π}{p_0} (p_1 + p_2)\right),
\]

where \(p_0\) is a constant with dimensions of momentum. These 18 observables are examples of modular variables which have played a distinguished role in the interpretation of the Aharonov-Bohm and related effects [29], the study of the Greenberger-Horne-Zeilinger proof for continuous variables [30], and in dynamic quantum nonlocality [31].

The 18 observables (14) comprise six sets of six mutually compatible observables. For instance, the six observables in

\[
\langle ABC \rangle = \langle αβγ \rangle = \langle Aαα \rangle = \langle Bββ \rangle = 1.
\]

Therefore, according to QM,
Interesting, observables $C$, $C''$, $C'$, $c''$, $c'$, and $y''$ are also compatible. This property can be derived from the following argument. Standard canonical commutation relations (13) imply Weyl’s canonical commutation relations (see, e.g., [32]),

$$\exp\left(-\frac{i}{\hbar}r_xi\right) \exp\left(-\frac{i}{\hbar}tp_i\right) = \exp\left(-\frac{i}{\hbar}rt\right) \exp\left(-\frac{i}{\hbar}tp_i\right) \exp\left(-\frac{i}{\hbar}r_xi\right).$$  \hspace{1cm} (17)

An important particular instance of these relations is obtained when

$$rt = \pm 2\pi\hbar.$$  \hspace{1cm} (18)

In this case, (17) reduces to

$$\exp\left(-\frac{i}{\hbar}r_xi\right) \exp\left(-\frac{i}{\hbar}tp_i\right) = \exp\left(-\frac{i}{\hbar}tp_i\right) \exp\left(-\frac{i}{\hbar}r_xi\right).$$  \hspace{1cm} (19)

From (18) and (19), it follows that

$$\begin{bmatrix} \cos\left(\frac{r}{\hbar}x_i\right), \cos\left(\frac{t}{\hbar}p_i\right) \end{bmatrix} = 0,$$  \hspace{1cm} (20a)

$$\begin{bmatrix} \sin\left(\frac{r}{\hbar}x_i\right), \sin\left(\frac{t}{\hbar}p_i\right) \end{bmatrix} = 0.$$  \hspace{1cm} (20b)

By the same token, the relation

$$\frac{1}{\sqrt{2}}\begin{bmatrix} x_1 + \frac{\hbar\pi}{p_0} p_2 \end{bmatrix}, \frac{1}{\sqrt{2}}\begin{bmatrix} -\frac{p_0^2}{\hbar^2}x_2 + p_1 \end{bmatrix} = i\hbar$$  \hspace{1cm} (21)

implies that $[C',c'] = 0$. The commutativity between the rest of the operators in the set $C'$, $C''$, $c'$, $c''$, $y'$, and $y''$ can be deduced in a similar way. The significance for fundamental issues in quantum mechanics of the fact that appropriate trigonometric functions of two observables may commute even if these observables do not was first pointed out in [29]. The common eigenbasis associated with each of the sets of commuting operators considered here are described in the Appendix.

Now let us calculate

$$C_{C'Y} = \exp\left[-i\left(\frac{p_0}{\hbar}x_1 + \frac{\pi}{p_0} p_2\right)\right] \exp\left[i\left(\frac{p_0}{\hbar}x_2 - \frac{\pi}{p_0} p_1\right)\right] \exp\left\{i\left[\frac{p_0}{\hbar}(x_1 - x_2) + \frac{\pi}{p_0}(p_1 + p_2)\right]\right\}.$$  \hspace{1cm} (22)

Since $[x_1,p_2] = 0$, $[x_2,p_1] = 0$, $[x_1 - x_2, p_1 + p_2] = 0$, $[x_1,x_2] = 0$, and $[p_1,p_2] = 0$, the exponentials on the right-hand side of (22) can be factorized as

$$\exp\left(-\frac{i}{\hbar}p_0 x_1\right) \exp\left(-i\frac{\pi}{p_0} p_2\right) \exp\left(i\frac{p_0}{\hbar} x_2\right) \exp\left(-i\frac{\pi}{p_0} p_1\right) \exp\left(i\frac{p_0}{\hbar} x_1\right) \exp\left(-i\frac{\pi}{p_0} x_2\right) \exp\left(i\frac{\pi}{p_0} p_1\right) \exp\left(i\frac{\pi}{p_0} p_2\right).$$  \hspace{1cm} (23)

Now we repeatedly apply another special instance of Weyl’s relations (17),

$$\exp\left(-\frac{i}{\hbar}r_xi\right) \exp\left(-i\frac{t}{p_1}\right) = -\exp\left(-i\frac{t}{p_1}\right) \exp\left(-\frac{i}{\hbar}r_xi\right),$$  \hspace{1cm} (24)

corresponding to

$$rt = \pm \pi\hbar.$$  \hspace{1cm} (25)

Using three times (25) and (24) in (23), we obtain

$$C_{C'Y} = -1.$$  \hspace{1cm} (26)

Consequently, according to QM,

$$\langle C_{C'Y} \rangle = -1.$$  \hspace{1cm} (27)

Therefore, from (16) and (27), the quantum-mechanical prediction for $|S|$ is

$$|\langle S_{QM} \rangle| = 6,$$  \hspace{1cm} (28)

which violates the upper bound of inequality (4), $|\langle S \rangle| \leq \sqrt{3}/3 = 5.19$. This violation is maximal and is the same for any state of the system, even for mixed states regardless their degree of mixture. The state independency of the violation is particularly interesting for continuous-variable systems where it is generally difficult to prepare specific states.

Equations (15) and (26) indicate that the nine unitary operators $A = A' + iA''$, $\ldots$, $Y = \gamma' + i\gamma''$ formally behave like those of the celebrated Peres-Mermin square of two-qubit operators [33,34]. A similar observation was made by Clifton [35].

**IV. CONCLUSIONS**

No fundamental difficulty seems to exist to observe a state-independent violation of noncontextual inequalities with continuous variables. For any quantum system admitting two continuous position observables, $x_1$ and $x_2$, and the corresponding canonically conjugate momenta, $p_1$ and $p_2$, we have shown that there exists a set of universal observables with continuous spectra which can experimentally reveal state-independent quantum contextuality. The observables $x_1$, $x_2$, $p_1$, and $p_2$ could represent, for instance, the position and momentum of a particle moving in a two-dimensional space, or the positions and momenta of two particles, each of them moving in a one-dimensional space [36], or the quadrature amplitudes of two modes of the electromagnetic field [37]. The required measurements, although discussed long ago in the literature, are probably hard to implement in actual experiments using specific physical systems, and this issue demands further research. But the important point is that, according to QM, there is no fundamental obstacle to carry out these measurements and observe the effect.

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**APPENDIX**

In this Appendix we provide the common eigenbasis associated with each of the sets of commuting operators considered in the paper. Following standard, self-explanatory notation,
we denote by $|x_{1,2}\rangle$ and $|p_{1,2}\rangle$ the eigenstates of the position and momentum operators $x_{1,2}$ and $p_{1,2}$, respectively. These eigenstates are normalized in the standard way: $\langle x_{1}'|x_{1}\rangle = \delta(x_{1} - x_{1}')$, etc. It is plain that the common eigenbasis corresponding to the sets $\{A,B,C\}$, $\{a,b,c\}$, $\{A,a,a\}$, and $\{B,b,b\}$ consist of states (again using self-explanatory notation) $\{|p_{1,2}|x_{1,2}\rangle$, $\{|p_{1,2}|x_{1,2}\rangle$, and $\{|p_{1,2}|p_{1,2}\rangle$, respectively. The common eigenbasis of the compatible operators $(a,b,\gamma)$ is constituted by states of the form $\{|x_{2,1}\rangle|p_{+}\rangle$, where $|x_{2,1}\rangle$ and $|p_{+}\rangle$ stand for eigenstates of the operators $x_{+} = x_{2} - x_{1}$ and $p_{+} = p_{1} + p_{2}$, respectively. If we interpret $x_{1}$ and $x_{2}$ as the coordinates of two particles of equal mass, then $x_{-} = x_{2} - x_{1}$ and $x_{+} = \frac{1}{2}(x_{1} + x_{2})$ represent the relative and center-of-mass coordinates, and $p_{-} = \frac{1}{2}(p_{1} - p_{2})$ and $p_{+} = p_{1} + p_{2}$ the concomitant canonically conjugate momenta.

Let us now consider the common eigenbasis of the set of operators $\{C,\gamma,\gamma\}$. In order to construct this eigenbasis, it is convenient to first introduce the observables

$$V_{1} = x_{1} + \frac{\pi \hbar}{p_{0}} p_{2}, \quad W_{1} = -\frac{p_{0}^{2}}{2 \pi} \hbar v_{2} + \frac{P_{1}}{2}, \quad (A1a)$$

$$V_{2} = x_{2} + \frac{\pi \hbar}{p_{0}} p_{1}, \quad W_{2} = -\frac{p_{0}^{2}}{2 \pi} \hbar x_{1} + \frac{P_{2}}{2}. \quad (A1b)$$

These observables satisfy the commutation relations, $[V_{1}, W_{k}] = i \hbar \delta_{jk}$, $[V_{j}, W_{1}] = [W_{j}, W_{k}] = 0$, for $j, k = 1, 2$. That is, the observables $V_{1,2}$ and $W_{1,2}$ comply with the same commutation relations as $x_{1,2}$ and $p_{1,2}$. In the standard $(x_{1,2})$ coordinate representation, the common eigenstates $|v_{1,2}\rangle$ of $V_{1}$ and $V_{2}$ are given by the wave function

$$\langle x_{1,2}|v_{1,2}\rangle = \frac{1}{2\pi} \exp \left[ i(v_{2}x_{1} + v_{1}x_{2}) - \frac{p_{0}^{2}}{2 \pi \hbar^{2}} x_{1} x_{2} \right], \quad (A2)$$

complying with the normalization condition $\langle v_{1,2}|v_{1,2}\rangle = \delta(v_{1} - v_{1}')\delta(v_{2} - v_{2}')$. The eigenvalues corresponding to the observables $V_{1,2}$ associated with eigenstate (A2) are $\frac{\pi \hbar^{2}}{p_{0}} v_{1,2}$. The eigenstates $|v_{1,2}\rangle$ constitute a (continuous) basis for the Hilbert space describing the system under consideration.

Now we can express the operators $C$ and $c$ in terms of $V_{1}$ and $W_{1}$,

$$C = \exp \left( -\frac{i}{\hbar} p_{0} V_{1} \right), \quad (A3a)$$

$$c = \exp \left( -\frac{i}{\hbar} \frac{2\pi \hbar}{p_{0}} W_{1} \right). \quad (A3b)$$

It follows from the commutation relation verified by the observables $V_{j}$ and $W_{j}$ and from (A3b) that the operator $c$ represents a displacement in the “direction” $v_{1}$,

$$c|v_{1},v_{2}\rangle = |v_{1} + \frac{2p_{0}}{\hbar} v_{2}\rangle. \quad (A4)$$

The commutation relations verified by the operators $V_{j}$ and $W_{j}$ and, in particular, relation (A4) imply that the common eigenbasis of the commuting operators $C$, $c$, and $V_{2}$ is given by the states

$$|\kappa,\epsilon,v_{2}\rangle = \frac{1}{\sqrt{2\pi}} \sum_{n=-\infty}^{\infty} \exp[i\kappa n] \frac{2p_{0}}{\hbar} n v_{2}, \quad (A5)$$

where $\kappa \in [0,2\pi)$, $\epsilon \in [0,2p_{0}/\hbar)$, and $-\infty < v_{1} < +\infty$. The states $|\epsilon + \frac{p_{0}^{2}}{\hbar^{2}} n, v_{2}\rangle$ in (A5) are common eigenstates of $V_{1}$ and $V_{2}$, given by wave functions of the form (A2). The eigenvalues of the eigenstate (A5) associated with the operators $C$, $c$, and $V_{2}$ are, respectively,

$$\exp \left( -i\kappa \frac{\pi \hbar v_{2}}{p_{0}} \right), \quad \exp \left( -i\kappa \frac{\pi \hbar v_{2}}{p_{0}} \right). \quad (A6)$$

The eigenstates (A5) are normalized as

$$\langle \kappa,\epsilon,v_{2}|\kappa',\epsilon',v_{2}'\rangle = \delta(\kappa' - \kappa')\delta(\epsilon' - \epsilon)\delta(v_{2} - v_{2}'). \quad (A7)$$

Since the operators $C$, $c$, and $\gamma$ satisfy the relation $\gamma C = -\frac{\pi}{2}$ [which can be derived in the same way as (26)], it follows that the state $|\kappa,\epsilon,v_{2}\rangle$ is also an eigenstate of the operator $\gamma$, with eigenvalue

$$-\exp \left( i \left( \frac{\pi \hbar v_{2}}{p_{0}} \right) \right). \quad (A8)$$

We have shown that the states $|\kappa,\epsilon,v_{2}\rangle$ constitute a common eigenbasis of the operators $C$, $c$, and $\gamma$. It is easy to verify that those states are also eigenstates of the six compatible observables, $C'$, $C''$, $c'$, $c''$, $\gamma'$, and $\gamma''$, with eigenvalues $\cos \left( \frac{\pi \hbar v_{2}}{p_{0}} \right)$, $-\sin \left( \frac{\pi \hbar v_{2}}{p_{0}} \right)$, $\cos(\kappa)$, $-\sin(\kappa)$, $-\cos(\kappa + \frac{\pi \hbar v_{2}}{p_{0}})$, and $-\sin(\kappa + \frac{\pi \hbar v_{2}}{p_{0}})$, respectively. Finally, it can be easily seen that the orthonormal states (A5) constitute a basis of the Hilbert space of the system under study. To see that, it is enough to verify that the states $|v_{1},v_{2}\rangle$ given by the wave functions (A2) (which clearly constitute a basis) can be expressed as linear combinations of the states (A5). Indeed, if we set $\epsilon = v_{1} - \frac{2p_{0}}{\hbar} m$, with $m$ equal to the integer part of $\frac{\hbar v_{1}}{2p_{0}}$,

$$m = \text{int} \left( \frac{\hbar v_{1}}{2p_{0}} \right), \quad (A9)$$

we have

$$|v_{1},v_{2}\rangle = \int_{0}^{2\pi} \frac{d\kappa}{2\pi} e^{-i\kappa m} |\kappa,\epsilon,v_{2}\rangle. \quad (A10)$$


