Experimental Fully Contextual Correlations

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Quantum correlations are contextual yet, in general, nothing prevents the existence of even more contextual correlations. We identify and test a noncontextuality inequality in which the quantum violation cannot be improved by any hypothetical postquantum theory, and use it to experimentally obtain correlations in which the fraction of noncontextual correlations is less than 0.06. Our correlations are experimentally generated from the results of sequential compatible tests on a four-state quantum system encoded in the polarization and path of a single photon.

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Introduction.—Quantum contextuality [1–3] refers to the fact that the predictions of quantum mechanics (QM) cannot be reproduced assuming noncontextuality of results (i.e., that the results are predefined and independent of other compatible tests) or, equivalently, noncontextual hidden variable theories. By compatible tests we mean those satisfying the following theory-independent definition: “If a physical system is prepared in such a way that the result of test $x_i$ is predictable and repeatable, and if a compatible test $x_j$ is then performed (instead of test $x_j$) a subsequent execution of test $x_i$ shall yield the same result as if test $x_j$ had not been performed” [4] (see [5] for other definitions of compatibility). In QM, two tests represented by self-adjoint operators $A$ and $B$ are compatible when $A$ and $B$ commute. This guarantees that the quantum predictions for compatible tests are given by a single probability measure on a single probability space. Compatibility implies that the probability $P(a_i|x_i)$ of obtaining the result $a_i$ for the test $x_i$ is independent of other compatible tests $x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n$, i.e.,

$$P(a_i|x_i) = \sum_{a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_n} P(a_1, \ldots, a_n|x_1, \ldots, x_n),$$

for all sets $x_1, \ldots, x_n$ of compatible tests, and where $P(a_1, \ldots, a_n|x_1, \ldots, x_n)$ is the joint probability of obtaining the results $a_1, \ldots, a_n$ for the compatible tests $x_1, \ldots, x_n$, respectively. Assumption (1) is formally equivalent to the no-signaling principle, but involves compatible tests instead of spacelike separated tests.

The assumption of the noncontextuality of results states that the result $a_i$ of test $x_i$ is the same regardless of other compatible tests being performed; it only depends on $x_i$ and some hidden variables $\lambda$. This implies that the correlation among the results of compatible tests can be expressed as

$$P(a_1, \ldots, a_n|x_1, \ldots, x_n) = \sum_{\lambda} \prod_{i=1}^{n} P(a_i|x_i, \lambda),$$

for some common distribution $P(\lambda)$.

Noncontextuality inequalities are expressions of the form

$$S = \sum T_{a_1, \ldots, a_n, x_1, \ldots, x_n} P(a_1, \ldots, a_n|x_1, \ldots, x_n) \leq_{\text{NC}} \Omega_{\text{NC}},$$

where $T_{a_1, \ldots, a_n, x_1, \ldots, x_n}$ are real numbers and $\leq_{\text{NC}} \Omega_{\text{NC}}$ denotes that the maximum value of $S$ for any noncontextual correlations [therefore satisfying (2)] is $\Omega_{\text{NC}}$. Quantum contextuality is experimentally observed through the violation of noncontextuality inequalities [6–9].

Quantum nonlocality [10] is a particular form of quantum contextuality which occurs when the tests are not only compatible but also spacelike separated. In this case, noncontextuality inequalities are called Bell inequalities [10]. In addition to applications such as device-independent quantum key distribution [11,12] and random number generation [13], which require spacelike separation, quantum contextuality also offers advantages in scenarios without spacelike separation. Examples are communication complexity [14], parity-oblivious multiplexing [15], zero-error classical communication [16], and quantum cryptography secure against specific attacks [17,18].

The goal of this work is to identify and perform an experiment with sequential quantum compatible tests, which produces correlations with the largest contextuality allowed under the assumption (1), which is assumed to be valid also for postquantum theories. For this purpose, we first introduce a measure of contextuality of the correlations, the noncontextual content $W_{\text{NC}}$, so that $W_{\text{NC}} = 0$ corresponds to the maximum contextuality. Then, we show how to experimentally obtain testable upper bounds to $W_{\text{NC}}$. Next, we show how graph theory allows us...
to identify experiments in which the upper bound to $W_{NC}$ predicted by QM is zero, and apply this method to single out an experiment for which $W_{NC} = 0$. Finally, we perform this experiment and obtain correlations in which $W_{NC} < 0.06$.

Noncontextual content.—Every correlation among compatible tests [therefore satisfying (1)] can be expressed as

$$P(a_1, \ldots, a_n|x_1, \ldots, x_n) = w_{NC}P_C(a_1, \ldots, a_n|x_1, \ldots, x_n) + (1 - w_{NC})P_C(a_1, \ldots, a_n|x_1, \ldots, x_n),$$

(4)

where $0 \leq w_{NC} \leq 1$, $P_{NC}(a_1, \ldots, a_n|x_1, \ldots, x_n)$ can be expressed as (2), and $P_C(a_1, \ldots, a_n|x_1, \ldots, x_n)$ satisfies (1) but cannot be expressed as (2). We define the noncontextual content $W_{NC}$ of the correlations as the maximum value of $w_{NC}$ over all possible decompositions as (4), i.e.,

$$W_{NC} = \max_{\{P_{NC}, P_C\}} w_{NC}.$$  

(5)

This definition is parallel to the definition of local content introduced in [19]. In fact, for correlations generated through spacelike separated tests, the noncontextual content equals the local content.

$\Omega_{NC}$, $\Omega_Q$, and $\Omega_C$ will denote, respectively, the maximum value of $S$ for noncontextual correlations [i.e., which can be expressed as (2)], quantum correlations, and correlations satisfying (1). Now consider correlations satisfying (1) and saturating $\Omega_Q$. Then, given a decomposition of such correlations as (4), with $w_{NC} = W_{NC}$, $\Omega_Q$ can be expressed as $\Omega_Q = \sum T_{a_1, \ldots, a_n, x_1, \ldots, x_n}[W_{NC}P_{NC}(a_1, \ldots, a_n|x_1, \ldots, x_n) + (1 - W_{NC})P_C(a_1, \ldots, a_n|x_1, \ldots, x_n)] = W_{NC}\sum T_{a_1, \ldots, a_n, x_1, \ldots, x_n} \times P_{NC}(a_1, \ldots, a_n|x_1, \ldots, x_n) + (1 - W_{NC})\sum T_{a_1, \ldots, a_n, x_1, \ldots, x_n} \times P_C(a_1, \ldots, a_n|x_1, \ldots, x_n)$. The first sum can be expressed in a noncontextual form, so it is upper bounded by $W_{NC}$. The second sum cannot be expressed in a noncontextual form, so it can only be upper bounded by $\Omega_C$. Hence, $\Omega_Q \leq W_{NC}\Omega_{NC} + (1 - W_{NC})\Omega_C$, and, taking into account that $\Omega_{NC} \leq \Omega_Q \leq \Omega_C$, then

$$W_{NC} \leq \frac{\Omega_C - \Omega_Q}{\Omega_C - \Omega_{NC}}.$$  

(6)

Any experimental violation $S_{\text{exp}}$ of a noncontextuality inequality indicates that $\Omega_C > \Omega_{NC}$ and, therefore, provides an upper bound on $W_{NC}$, namely $W_{NC} \leq (\Omega_C - S_{\text{exp}})/(\Omega_C - \Omega_{NC})$. Assuming that the maximum $S_{\text{exp}}$ in an ideal experiment is given by $\Omega_Q$, to observe correlations with zero noncontextual content, here called fully contextual correlations, one has to test a noncontextuality inequality such that its maximum quantum violation equals its maximum possible violation under the assumption (1), i.e., an inequality for which $\Omega_{NC} < \Omega_Q = \Omega_C$.

However, even if $\Omega_Q = \Omega_C$, inherent imperfections of actual experiments will prevent the observation of $W_{NC} = 0$. In general, the more complex the experiment to produce the required quantum correlations is, the higher the probability that experimental imperfections lead to a higher upper bound for the noncontextual content. Therefore, the task is to identify the simplest noncontextuality inequality violated by QM and such that $\Omega_{NC} = \Omega_C$.

Graph approach.—We addressed this problem by using a connection between graph theory and noncontextuality inequalities noticed in [20]: For any graph there is a noncontextuality inequality for which $\Omega_{NC}$, $\Omega_Q$, and $\Omega_C$ are given, respectively, by the independence number, the Lovász number, and the fractional packing number of the graph [21]. We calculated these three numbers for all nonisomorphic graphs with less than 11 vertices, and found that there are no graphs with less than 10 vertices with $\Omega_{NC} < \Omega_Q = \Omega_C$, and there are only four 10-vertex graphs with these properties [21]. The maximum quantum violation of noncontextuality inequalities associated with three of them requires quantum systems of dimension higher than four, while dimension four is enough for the graph in Fig. 1. The inequality associated with the graph is constructed by looking for propositions involving compatible tests, such that each vertex represents one proposition in the inequality and the edges only link propositions that cannot be simultaneously true. Then, the inequality is simply given by the sum of all the probabilities of the propositions represented in the graph.

For the graph in Fig. 1, it can be easily seen that the following noncontextuality inequality is in one-to-one correspondence with the graph:

$$S = P(010|012) + P(111|012) + P(011|02) + P(000|03)$$

+ $P(111|03) + P(000|14) + P(011|25) + P(010|345)$

+ $P(111|345) + P(10|35) \leq_{NC} 3,$

(7)

where $P(10|35)$ is the probability of obtaining result 1 when test 3 is performed and result 0 when test 5 is performed. In this case, the coefficients $T_{a_1, \ldots, a_n, x_1, \ldots, x_n}$ in (3) are all 1. The noncontextual bound, $\Omega_{NC} = 3$, can be obtained from the independence number of the graph in

FIG. 1. Graph corresponding to inequality (7). Vertices represent propositions. For example, 01|25 means “result 0 is obtained when test 2 is performed, and result 1 is obtained when test 5 is performed.” Edges link propositions that cannot be simultaneously true. For example, 01|25 and 01|02 are linked, since in the first proposition the result of test 2 is 0, while in the second proposition the result of test 2 is 1.
Fig. 1. The maximum quantum violation of inequality (7) and its maximum possible violation under the assumption (1) can be obtained from the Lovász and the fractional packing numbers of the graph in Fig. 1, respectively [21]. This gives

$$\Omega_Q = \Omega_C = 3.5.$$  

(8)

The maximum quantum violation can be achieved by preparing a four-state quantum system in the state

$$|\psi\rangle = \frac{1}{\sqrt{2}} (|0\rangle + |3\rangle),$$  

(9)

where $\langle 0 | = (1, 0, 0, 0)$, $\langle 1 | = (0, 1, 0, 0)$, $\langle 2 | = (0, 0, 1, 0)$, and $\langle 3 | = (0, 0, 0, 1)$, and with the tests represented by the following tensor products of Pauli matrices $\sigma_i$ and the $2 \times 2$ identity matrix $I$:

$$0 = \sigma_x \otimes 1, \quad 1 = 1 \otimes \sigma_z, \quad 2 = \sigma_y \otimes \sigma_z, \quad 3 = 1 \otimes \sigma_x, \quad 4 = \sigma_z \otimes 1, \quad 5 = \sigma_z \otimes \sigma_x.$$  

(10)

The results 0 and 1 correspond to the eigenvalues $-1$ and $+1$, respectively, of the operators in (10). Notice that every probability in (7) includes only pairs or trios of mutually compatible tests.

Experiment.—The experiment required two-test sequences [for instance, to obtain $P(00)[14]$], and three-test sequences [for instance, to obtain $P(010)[012]$]. We built six devices for the six dichotomic tests defined in (10). The sequential tests were performed using cascade setups [9] like the one shown in Fig. 2. We tested inequality (7) using the spatial path and polarization of a single photon carrying a four-state quantum system with the following encoding:

$$|0\rangle = |t, H\rangle, \quad |1\rangle = |t, V\rangle, \quad |2\rangle = |r, H\rangle, \quad |3\rangle = |r, V\rangle.$$  

(11)

where $t$, $r$, $H$, and $V$ denote the transmitted path, reflected path, horizontal, and vertical polarization of the photon, respectively.

The cascade setup used to implement two sequential tests on a single photon consists of three parts: state preparation, testing devices, and detectors. The preparation of the polarization-spatial path-encoded single-photon state $|\psi\rangle$ is achieved using a source of $H$-polarized single photons. This single-photon source consists of an attenuated stabilized narrow bandwidth diode laser emitting at the wavelength of 780 nm. This laser offers a long coherence length. The two-photon coincidences were set to a negligible level by attenuating the laser to a mean photon number of 0.06 per time coincidence window. This source is followed by a half-wave plate (HWP) set at 22.5° and a polarizing beam splitter (PBS), allowing the photon to be distributed with equal probability between the two paths $t$ and $r$ with the right polarization $H$ and $V$, respectively [see Fig. 2].

Then, the photon in the two paths enters the device for testing $x_1$ through the device’s input and follows one of the two possible outputs, which correspond to the values $+1$ and $-1$. After each of the two outputs, we placed a device for testing $x_2$. We used two identical devices for testing $x_2$. Finally, we placed a single-photon detector ($D$) at the output of the two devices $x_2$. The same idea is used for sequences of three tests $x_1$, $x_2$, and $x_3$, by adding four devices for measuring $x_3$ and using eight single-photon detectors.

Devices for measuring the six tests defined in (10) are given in Fig. 3. Measurements 1 and 3 are standard polarization measurements using a PBS and a HWP which map the polarization eigenstate of the operator to $|t, H\rangle$ and $|r, V\rangle$. The mapping to the eigenstates of test 0, namely $(|t\rangle \pm |r\rangle)/\sqrt{2}$, was accomplished by interfering the two paths in a 50/50 beam splitter (BS). A wedge ($W$) is placed in one of the paths to set the phase between both paths [see Fig. 3]. Tests 2 and 5 are represented by the tensor product of a spatial path and a polarization operator so they have a four-dimensional eigenspace. However, since the tests need to be rowwise and columnwise compatible, only their common eigenstates can be used for distinguishing the eigenvalues. Measurement 4 requires us only to distinguish between paths $t$ and $r$. We needed to recreate the eigenstates of the performed tests after each mapping and before entering the next test, since our single-test devices map eigenstates to a fixed spatial path and polarization.

All interferometers in the experimental setup were based on a displaced Sagnac configuration. The stability of these interferometers is very high. We obtained visibilities over 99% for phase insensitive interferometers, and ranging between 90% and 95% for phase sensitive interferometers. We used silicon avalanche photodiodes calibrated to have...
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