Recursive proof of the Bell–Kochen–Specker theorem in any dimension $n > 3$

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Received 11 March 2005; accepted 23 March 2005
Available online 8 April 2005
Communicated by P.R. Holland

Abstract

We present a method to obtain sets of vectors proving the Bell–Kochen–Specker theorem in dimension $n$ from a similar set in dimension $d$ ($3 \leq d < n \leq 2d$). As an application of the method we find the smallest proofs known in dimension five (29 vectors), six (31) and seven (34), and different sets matching the current record (36) in dimension eight.

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PACS: 03.65.Ud
Keywords: Kochen–Specker theorem; Entanglement and quantum non-locality

1. Introduction

The Bell–Kochen–Specker (BKS) theorem $^{[1,2]}$ states that quantum mechanics (QM) cannot be simulated by non-contextual hidden-variable theories. Any hidden-variable theory reproducing the predictions of QM must be contextual in the sense that the result of an experiment must depend on which other compatible experiments are performed jointly. The BKS theorem is independent of the state of the system, and is valid for systems described in QM by Hilbert spaces of dimension $d \geq 3$.

A proof of the BKS theorem consists of a set of physical yes–no tests, represented in QM by one-dimensional projectors, to which the rules of QM do not allow the assignment of predefined “yes” or “no” answers, regardless of how the system was prepared. In this Letter, yes–no tests will be represented by the vectors onto which the projectors project.

Several proofs of the BKS theorem in dimensions three, four and eight are known: see, for instance, $^{[3]}$ and the references in $^{[4]}$. General procedures for extending the demonstration to a finite dimension $n$ also
exist [4–6]. In Section 2, we present a new method to obtain sets of vectors proving the BKS theorem in dimension \( n \) from a similar set in dimension \( d (3 \leq d < n \leq 2d) \). In Section 3 we compare this method with those of [4–6]. The main interest of this method is that it leads to the smallest proofs known in dimension five (29 vectors), six (31) and seven (34), and to different sets matching the current record (36) in dimension eight. These proofs are explicitly presented for the first time in Section 4; a preliminary version of them was referred to in [7,8].

Which one is the smallest number of yes–no tests needed to prove the BKS theorem in each dimension? This is an old question [4]. Recently, it has been proven that the answer is 18 for dimension four [9], and that there are no proofs with less yes–no tests in any dimension [10]. The proofs presented in Section 4 give an upper bound to this search in dimensions five to eight. The important point is that these bounds are sufficiently small so as to apply recently developed approaches capable to exhaustively explore all possible proofs of the BKS theorem [9,10]. The practical limitation of these approaches is that the complexity of the exploration grows exponentially with the number of vectors, making it difficult to explore all possible sets involving 30 vectors or more.

A set of \( n \)-dimensional vectors \( X := \{ u_j \}_{j=1}^N \) is a proof of the BKS theorem if we cannot assign to each vector \( u_j \) a \( v(u_j) \) such that:

(a) Each \( v(u_j) \) has a uniquely defined value, 0 or 1 (“black” or “white”); this value is non-contextual, i.e., does not depend on which others \( v(u_k) \) are jointly considered.

(b) \( \sum_{i=1}^n v(u_i) = 1 \) V set of \( n \) mutually orthogonal vectors \( \{ u_i \}_{i=1}^n \in X \).

In that case \( X \) is said to be “non-colourable”. A proof of the BKS theorem is said to be “critical” if all vectors involved are essential for the proof.

2. Recursive proof of the Bell–Kochen–Specker theorem

Let \( A := \{ a_i \}_{i=1}^f \), \( a_i := (a_{i1}, \ldots, a_{id}) \), be a proof in dimension \( d \). For any \( n := d + m, 1 \leq m \leq d \), let us define two sets of \( n \)-dimensional vectors, \( B^* := \{ b_i \}_{i=1}^f \), \( C^* := \{ c_i \}_{i=1}^f \), obtained by appending to each vector \( a_i \) \( m \) zero components on the right and on the left, respectively; \( b_i := (a_{i1}, \ldots, a_{id}, 0, \ldots, 0) \), \( c_i := (0, \ldots, 0, a_{i1}, \ldots, a_{id}) \). Let us also define the following sets of \( n \)-dimensional vectors: \( \bar{B} := \{ b_j \}_{j=f+1}^{f+m} \), \( b_{jk} := \delta_{j-f+d,k} \), \( \bar{C} := \{ c_j \}_{j=f+1}^{f+m} \), \( c_{jk} := \delta_{j-f,k} \), \( \bar{B} := B^* \cup \bar{B} = \{ b_j \}_{j=1}^{f+m}, C := C^* \cup \bar{C} = \{ c_j \}_{j=1}^{f+m} \).

Lemma. \( B \) is BKS-colourable if and only if

\[
\sum_{j=f+1}^{f+m} v(b_j) = 1.
\]

Proof. The sets of \( d \) mutually orthogonal vectors in \( A \) become sets of \( n \) mutually orthogonal vectors in \( B \), sharing the last \( m \) vectors, \( b_j \in \bar{B}, j = f + 1, \ldots, f + m \). If condition (1) is fulfilled, we can colour \( B \) simply by assigning the values \( v(b_j) = 0, j = f + 1, \ldots, f + m \); the impossibility to colour set \( A \) in dimension \( d \) following rules (a), (b) implies the impossibility to colour \( B \) in dimension \( n \).

The same reasoning applies to \( C \): \( C \) is colourable if and only if

\[
\sum_{j=1}^m v(c_j) = 1.
\]

Theorem. \( D := B \cup C \) is a non-colourable set.

Proof. If \( d < n \leq 2d \), then \( \bar{B} \cap \bar{C} = \emptyset \); conditions (1) and (2), necessary to colour \( B \) and \( C \), would imply the existence of two mutually orthogonal vectors, \( b_k \in \bar{B}, c_l \in \bar{C} \), with values \( v(b_k) = 1, v(c_l) = 1 \); this prevents \( D = B \cup C \) from being coloured following rule (b); therefore \( D \) is a non-colourable set.
3. Comparison with other methods

In the following, we will present some outcomes of the \("n \leq 2d\) method" introduced in Section 3, and compare them with those obtained with other methods.

The \(n \leq 2d\) method allows us to construct non-colourable sets in any dimension \(n \geq 4\), starting with one non-colourable set in dimension three (first in dimension four, five, and six; then up to dimension 12 using the sets generated in this first step, etc.): we could start with Conway and Kochen’s 31-vector critical set, reviewed in [12], or with Peres’ 33-vector critical set [13]; nevertheless, smaller sets in dimension \(n \geq 5\) can be obtained starting from suitable non-colourable sets in dimension \(n = 4\) [3,13]. In general, starting with a \(f\)-vector non-colourable set in dimension \(d\), this method produces non-colourable sets with at most \(g \leq 2^d(f + kd)\) vectors in dimension \(2kd\).

The first row in Table 1 shows the number of vectors of the proofs in dimensions five to eight obtained by several methods, in parenthesis are the sizes of the smallest critical subsets obtained by computer (see Section 4 for examples of such sets). Records in each dimension are in boldface. In dimension 4, starting with the 18-vector sets of Ref. [3], ZP’s method will produce \(2^n\) different critical sets with \(9 \times 2^{k+1}\) vectors in dimension \(n = 2^k+2\), \(k = 1, 2, \ldots\), compared with a non-critical \(g\)-vector non-colourable sets with \(9 \times 2^{k+1} < g \leq (9 + 2k) \times 2^{k+1}\) obtained by our previous \(n \leq 2d\) method.

The third row in Table 1 represents the number of vectors of the non-colourable sets obtained with the ZP method, starting with the smallest non-colourable sets currently known in three and four dimensions (Conway and Kochen’s 31-vector critical set [11], and any of the 18-vector critical sets of Ref. [3], respectively). The non-colourable sets in dimensions six and seven are larger than those previously discussed (and do not contain smaller non-colourable subsets, because they are critical). In dimension \(n = 8\), starting from couples of the 18-vector sets of Ref. [3] ZP’s method produces 256 different critical 36-vector sets (which, actually, are the same 256 record sets obtained with the \(n \leq 2d\) method).

Composing two non-colourable sets with \(f\) and \(g\) vectors in dimensions \(d\) and \(m\), Zimba and Penrose’s (ZP’s) method [5] produces a non-colourable set with \(f + g\) in dimension \(d + m\), which is critical if both components are critical. ZP’s method leaves out the case \(n = 5\), since both \(d\) and \(m\) must be greater or equal than three. The number of vectors increases linearly if the dimension is a multiple of the initial one; starting with a \(f\)-vector critical set in dimension \(d\), ZP’s method produces critical sets with \(hf\) vectors in dimension \(hd\) (note that, although these sets do not contain any subset that is also a non-colourable set, the number \(hf\) is only an upper bound to the size of the smallest possible non-colourable sets in dimension \(hd\)). In particular, starting from the 18-vector sets of Ref. [3], ZP’s method will produce \(2^n\) different critical sets with \(9 \times 2^{k+1}\) vectors in dimension \(n = 2^k+2\), \(k = 1, 2, \ldots\), compared with a non-critical \(g\)-vector non-colourable sets with \(9 \times 2^{k+1} < g \leq (9 + 2k) \times 2^{k+1}\) obtained by our previous \(n \leq 2d\) method.

Other methods produce larger non-colourable sets: for instance, in [15] Peres constructs a proof in di-

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**Table 1**

<table>
<thead>
<tr>
<th>Dimension (n)</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td>(n \leq 2d) method, starting from Peres’ 24-vector set in (d = 4) [13]</td>
<td>39(^{(29)})</td>
<td>44(^{(31)})</td>
<td>47(^{(34)})</td>
<td>48(^{(36)})</td>
</tr>
<tr>
<td>(n \leq 2d) method, starting from the 18-vector critical set (S_4) in (d = 4)</td>
<td>31(^{(29)})</td>
<td>35(^{(32)})</td>
<td>37(^{(34)})</td>
<td>38(^{(36)})</td>
</tr>
<tr>
<td>Zimba–Penrose method [5], using Conway and Kochen’s 31-vector critical set in (d = 3) [12] and (S_4)</td>
<td>\ldots</td>
<td>62</td>
<td>49</td>
<td>36</td>
</tr>
</tbody>
</table>

\(^{(29)}\) In parenthesis are the sizes of the smallest critical subsets obtained by computer (see Section 4 for examples of such sets). Records in each dimension are in boldface.
mension $d + 1$ starting from another in dimension $d$; only one initial non-colourable set is needed to reach recursively any dimension $n$, but the sizes of the non-colourable sets obtained increase rapidly in general, and a search for critical subsets is necessary in order to avoid very large sets. Finally, the method in Ref. [4] is a generalization to arbitrary dimension of the three-step original construction of Kochen and Specker’s [2], explicitly showing the relation between the different kinds of non-colourable sets and the three versions of the BKS theorem, but it is almost as inefficient as regards the number of vectors involved (96 in dimension $n = 3$, 136 in $n = 4$, and $35n$ if $n \geq 5$) as the original Kochen–Specker 117-vector proof in dimension $n = 3$ was.

4. Record critical proofs in dimensions four to eight

As an application of the method introduced in Section 2 we have obtained the smallest proofs known in dimension five (29 vectors), six (31) and seven (34), and several sets matching the current record (36) in dimension eight. We have omitted the normalization constants of the vectors in order to simplify the notation.

Let us start with an example [3] of the smallest proof of the BKS theorem, not only in dimension four [9], but in any dimension [10]: $S_3 := \{(1, 0, 0, 0), (0, 0, 1, 0), (0, 0, 0, 1), (1, 1, 0, 0), (0, 1, 1, 0), (0, 0, 1, 1), (1, -1, 0, 0), (0, 1, -1, 0), (1, 1, 0, 1), (0, 0, 1, 0), (0, 1, 0, -1), (1, 0, 0, 1), (1, -1, 1, -1), (0, 1, 1, -1), (1, -1, -1, 1), (1, 1, 1, -1), (1, 1, -1, 1), (-1, 1, 1, 1)\}$. $S_3$ is one of the 16 similar sets of Ref. [3]. We can construct 9 tetrads of mutually orthogonal vectors in terms of the 18 elements of $S_3$; each vector is orthogonal to 7 others in the set and appears in 2 tetrads. The non-colourability of the set is proved by a parity argument: the sum of values for each tetrad is 1, following (b); therefore the sum of the values in the 9 tetrads must be 9, but each value appears twice and therefore the sum is even, following (a). We have chosen a different set than in [3] because the proof in $n = 5$ deduced from that set has 33 vectors, instead of 31 as when starting from $S_3$.

An example of the smallest known proof in dimension five is $S_5 := \{(a, 0, 0), (0, a) : a \in S_4\} - \{(0, 1, 0, 0, 0), (0, 0, 1, 0, 0)\}$. We can construct 16 pentads of mutually orthogonal vectors in terms of the 29 elements of $S_5$. The non-colourability of $S_5$ can be proved by exhaustive computer tests (we have found no analytic proof of the non-colourability of this or the following $S_6$, $S_7$ sets). This is one of the 120 similar 29-vector critical subsets in the 39-vector non-colourable set obtained from P-24.

An example of the smallest known proof in dimension six is $S_6 := \{(a, 0, 0), (0, 0, a) : a \in S_5\} \cup \{(0, 1, 0, 0, 0, 0), (1, 0, 0, 0, 1, 0), (1, 1, 0, 0, 0), (0, 0, 0, 1, 0, 0), (0, 0, 0, 0, 0, 0), (0, 0, 1, -1, 0, 0), (1, -1, 1, 0, 0), (0, 1, 0, 1, 0)\}$; vectors in boldface appear when applying the $n \leq 2d$ method to P-24, but not to $S_6$. We can construct 16 hexads of mutually orthogonal vectors in terms of the 31 elements of $S_6$. The non-colourability of $S_6$ has been proved by computer. This is one of the 128 similar 31-vector critical subsets in the 44-vector non-colourable set obtained from P-24.

An example of the smallest known proof in dimension seven is $S_7 := \{(a, 0, 0, 0), (0, 0, 0, a) : a \in S_5\} - \{(0, 0, 0, 1, 0, 0, 0)\}$. We can construct 28 heptads of mutually orthogonal vectors in terms of the 34 elements of $S_7$. The non-colourability of $S_7$ has been proved by computer. This is one of the 144 similar 34-vector critical subsets in the 47-vector non-colourable set obtained from P-24.

An example of the smallest known proof in dimension eight [14] is $S_8 := \{(a, 0, 0, 0, 0), (0, 0, 0, 0, a) : a \in S_7\}$. We can construct 81 octads of mutually orthogonal vectors in terms of the 36 elements of $S_8$; each vector is orthogonal to 25 others in the set and belongs to 18 octads. The non-colourability of $S_8$ can be proved by a parity argument: the number of octads is odd, but each vector appears an even number of times (actually, the non-colourability of $S_8$ is a consequence of the non-colourability of $S_7$ and the application of the ZP method). $S_8$ is one of the 256 similar 36-vector critical subsets in the 48-vector non-colourable set obtained from P-24.

Acknowledgements

We would like to thank the late Asher Peres for his comments and advice.
References