How much larger quantum correlations are than classical ones

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Considering as distance between two two-party correlations the minimum number of half local results one party must toggle in order to turn one correlation into the other, we show that the volume of the set of physically obtainable correlations in the Einstein-Podolsky-Rosen-Bell scenario is $(3\pi/8)^2 \approx 1.388$ larger than the volume of the set of correlations obtainable in local deterministic or probabilistic hidden-variable theories, but is only $3\pi^2/32 \approx 0.925$ of the volume allowed by arbitrary causal (i.e., no-signaling) theories.

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I. INTRODUCTION

Quantum information (that is, information carried by microscopic systems described by quantum mechanics such as atoms or photons) can connect two spacelike separated observers by correlations that cannot be explained by classical communication. This fact, revealed by Bell’s inequalities and violations thereof [1–4], is behind common statements such as that quantum correlations are “stronger” or “larger” than classical ones, or that quantum-mechanical systems may be “further correlated” than those obeying classical physics [5], and has been described as “the most profound discovery of science” [6]. Given its fundamental importance, it is surprising that the question of how much “larger” than classical correlations quantum correlations are has not, to my knowledge, a precise answer beyond the fact that quantum mechanics violates Clauser-Horne-Shimony-Holt (CHSH) inequalities [2] up to $2\sqrt{2}$ (Tsirelson’s bound [7]), while the bound for local hidden-variable theories is just $2$ [2]. To be more specific, if by $Q$ we denote the set of all correlations allowed by quantum mechanics in a given experimental scenario, and by $C$ the corresponding set of correlations allowed by any local deterministic [1,2] or probabilistic [3,4] hidden-variable theory (the set of correlations allowed by both types of theories turns out to be the same [3]), a more precise measure of how much larger quantum correlations are when compared to classical ones would be the ratio between the volumes (i.e., the hyper-volumes or contents) of both sets, $V_Q/V_C$. To my knowledge, the value $V_Q/V_C$, even for the simplest nontrivial experimental scenario, cannot be found anywhere in the literature.

Moreover, another interesting problem is why quantum correlations cannot be even “larger” than they are. For instance, Popescu and Rohrlich raised the question of whether the no-signaling condition could restrict the set of physically obtainable correlations to those predicted by quantum mechanics [8]. Although they proved this conjecture to be false [8] (however, see [9]), a precise measurement of the ratio between the volume of quantum correlations and the volume of the set $L$ of all possible correlations allowed by any arbitrary causal (i.e., no-signaling) theory, $V_Q/V_L$, still cannot be found in the literature.

A. A natural distance in the space of correlations

The three spaces compared in this paper have quite different status:
- $Q$, the set of correlations obtainable by quantum mechanics, gives a perfect account of all preparations and measurements which are possible in nature.
- $C$, the set of correlations obtainable by local deterministic or probabilistic hidden-variable theories, does not give a complete account of all possible preparations and measurements.
- $L$, the set of correlations obtainable by any arbitrary causal (i.e., no-signaling) theory, is nonphysical, in the sense that it admits preparations and measurements that are impossible.

The volume of a set of correlations depends on the choice of a measure. In order to compare these three sets, the first problem to address is finding a common measure with a clear meaning for the three sets. Some criteria proposed in the literature are:

(i) The minimum amount of classical communication between the parties necessary to reproduce some correlations, assuming that the parties share correlations belonging to $C$ [10].

(ii) The number of trials of the experiment required to observe that a set of correlations provides a substantial violation of the predictions of $C$ [11].

(iii) The reduction of the amount of classical communication needed for some specific distributed computational task compared to the classical communication needed when assuming correlations belonging to $C$ [12].

Each of these proposals would lead to a different distance in the space of correlations, and all of these distances have their own problems. All of them privilege some or all the elements of $C$. In particular, (i) and (ii) would assign a volume zero to $C$, while (iii) would introduce a distance which would be sometimes negative.

In this paper we shall define as distance between two two-party correlations $(A_iB_j)$ and $(A_iB_i)$ the minimum number of half local results one party must toggle in order to turn one correlation into the other. For instance, suppose that the results of repeating Alice’s experiment $A_i$ and Bob’s experiment $B_j$ a million times are completely uncorrelated, i.e., $\langle A_iB_j \rangle = 0$. A way of increasing the correlation is that Alice toggles some of her results (−1 or +1) in order to be perfectly

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correlated with Bob’s corresponding results. If they want to
increase the correlation to 0.5, Alice must toggle at least
250,000 results or, equivalently, 0.25 results per single
experiment. Analogously, to change from \((A,B) = 0.7\) to 0.1,
Alice must toggle a minimum of 0.3 results per experiment.
As can be easily seen, this distance induces a uniform mea-
sure over the space of correlations.

Besides its clear physical meaning, this measure has some
other advantages. It only privileges the point where the re-
sults of the experiments of the two parties are completely
uncorrelated, what is reasonable from a physical perspective.
Moreover, the corresponding volumes have a simple inter-
pretation: the probability that four random numbers between
\(-1\) and \(+1\) belong to \(C\) (i.e., satisfy the CHSH inequalities)
is the volume of \(C\) divided by the volume of \(L\), etc. Conse-

quently, \(V_Q/V_C\) is a reasonable measure of how much larger
\(Q\) is than \(C\), and \(V_Q/V_C\) is a physically reasonable measure
of how much larger the set of correlations could be without
violating the no-signaling condition.

B. The Einstein-Podolsky-Rosen (EPR)-Bell scenario

The EPR-Bell scenario [1–4,13,14] is the simplest and
most basic one where the difference between classical and
quantum correlations arises. It consists of two alternative di-

dichotomic experiments (i.e., having only two possible out-
comes, which we can label \(\pm 1\)), \(A_0\) or \(A_1\), on a subsystem \(A\),
and another two alternative dichotomic experiments, \(B_0\) or
\(B_1\), on a distant subsystem \(B\). In this scenario the set of
correlations \((A,B)\) is four-dimensional. It is the standard sce-
nario in which the Einstein-Podolsky-Rosen (EPR) argument
[13] is presented [14] and in which violations of Bell’s in-

equalities [1–4] have been experimentally demonstrated
[15–17], and also the one in which recent experiments to
search for stronger-than-quantum correlations [18–21] has
been discussed. On the other hand, the EPR-Bell scenario is

contained in any experimental scenario involving more sub-

systems, more experiments per subsystem, or more discrete
outcomes per experiment.

Hereafter, we shall consider the EPR-Bell scenario and
will not make any assumptions about the type of physical
subsystems considered, their state before the local ex-

periments, or the type of local experiments performed.

The essential elements of the EPR-Bell scenario may be

summarized as two boxes, each with two possible inputs (the
two alternative local experiments) and two possible out-
comes [22]. Each possible pair of boxes can be characterized
by a set of \(2^4\) joint probabilities for the various possible
outcomes: \(P(A_i = a, B_j = b)\), \(a, b \in \{-1,1\}\). These proba-

bilities satisfy positivity (i.e., for all \(A_i, B_j\), and \(a, b \in \{-1,1\}\),
\(P(A_i = a, B_j = b) \geq 0\)) and a normalization condition
[i.e., for all \(A_i, B_j\), and \(a, b \in \{-1,1\}\), \(\sum_{a,b \in \{-1,1\}} P(A_i = a, B_j = b) = 1\)], and constitute (without taking into account

further constraints) a set of dimension 12 (of dimension 8 if
we impose the no-signaling constraint). The set of correla-
tions is a four-dimensional projection of the set of joint prob-
abilities. The connection between both sets is given by

\[
\langle A,B \rangle = \sum_{k,l \in \{-1,1\}} a_k b_l P(A_i = a, B_j = b),
\]

For the EPR-Bell scenario, and assuming the uniform
measure induced by the distance introduced in Sec. I A, in
Sec. II we calculate \(V_C\), in Sec. III we calculate \(V_Q\), and in
Sec. IV we calculate \(V_Q/V_C\) and \(V_Q/V_C\), and quantify the success of several previous attempts to charac-
terize \(Q\) by means of linear [7] and quadratic [23] Bell-like
inequalities. Finally, in Sec. V we suggest further lines of
research.

II. CORRELATIONS ALLOWED BY LOCAL
HIDDEN-VARIABLE THEORIES

Froissart [24] and Fine [25,26] (see also [27–28]) proved
that, for the EPR-Bell scenario, the set of all joint probabili-
ties attainable by local hidden-variable theories (i.e., theories
in which the local variables determine the probability distri-

bution for the different possible results of the local experi-
ments) is an eight-dimensional polytope with 16 vertices and
24 faces. The four-dimensional projection corresponding to
the set \(C\) of all correlations that can be attained by local
hidden-variable theories is defined by eight CHSH inequalities.
To be precise, a set of four real numbers \((A,B)\) \((i,j = 0,1)\)
belongs to \(C\), i.e., is attainable by local hidden-

variable theories, if and only if

\[
|\langle A_0 B_0 \rangle + \langle A_1 B_1 \rangle + \langle A_1 B_0 \rangle + \langle A_0 B_1 \rangle - 2 \langle A_0 B_1 \rangle| \leq 2,
\]

for all \(i,j = 0,1\). The volume of this four-dimensional set \(C\) can
be easily calculated.

\[
V_C = \frac{2^5}{3}.
\]

III. CORRELATIONS ALLOWED BY ARBITRARY
CAUSAL THEORIES

Let us consider theories where the only restriction is that
signaling is forbidden (i.e., the two distant observers cannot
signal to one another via their choice of input). The no-
signaling condition restricts the set of joint probabilities. The
no-signaling condition imposes that the marginal probabil-

ities \(P(B=b)\) \(P(A=a)\) should be independent of the choice
of \(A_i\) \(B_j\), for all \(B_j\) and \(b \in \{-1,1\}\) [for all \(A_i\) and \(a \in
\{-1,1\}\)]. This implies eight restrictions on the set of joint
probabilities:

\[
P(A_0 = 0, B_j = b) + P(A_0 = -1, B_j = b) = P(A_i = 1, B_j = b) + P(A_i = -1, B_j = b)
\]

and \(P(A_i = a, B_0 = 1) + P(A_i = a, B_0 = -1) = P(A_i = a, B_1 = 1) + P(A_i = a, B_1 = -1)\)
for all \(A_i\), \(B_j\), and \(a, b \in \{-1,1\}\), so that the set of all possible joint probabili-

ties satisfying the no-signaling condition has dimension 8.
This set is a convex polytope with 24 vertices and 16 faces
[29]. Most of the points of this set are not physically realizable;
however, its potential usage as an information theoretic
resource has been recently investigated [12].

However, the restrictions imposed on the set of joint
probabilities by the no-signaling condition do not imply new

nontrivial restrictions on the set of correlations \((A,B)\). There
are either sets of joint probabilities violating no-signaling but
satisfying inequalities (2), for instance, \(P(A_0 = 1, B_0 = 1) = P(A_0 = 1, B_0 = -1) = P(A_0 = -1, B_1 = 1) = P(A_0 = -1, B_1 =
012113-2
HOW MUCH LARGER QUANTUM CORRELATIONS ARE...

\[-1 = P(A_1 = 1, B_0 = 1) = P(A_1 = 1, B_0 = -1) = P(A_1 = -1, B_1 = 1) = P(A_1 = -1, B_1 = -1) = 1/2, \]

and sets satisfying no-signaling but maximally violating Eq. (2), for instance, \(P(A_0 = 1, B_0 = 0) = 1 = P(A_0 = 1, B_0 = -1) = P(A_0 = -1, B_0 = 1) = P(A_0 = -1, B_0 = -1) = P(A_1 = 1, B_1 = 1) = P(A_1 = -1, B_1 = -1) = P(A_1 = -1, B_1 = 1) = 1/2. \]

Therefore the set \([8]C\) of all correlations that can be attained by arbitrary causal theories is simply defined by the eight inequalities

\[
|A_jB_j| \leq 1, \quad (4)
\]

for \(i,j = 0,1\). \(C\) is a four-dimensional cube (a tesseract). Its volume is

\[
V_C = 2^4. \quad (5)
\]

Comparing Eqs. (3) and (5), we conclude that the volume of the set of correlations attainable by local hidden-variable theories is just 2/3 of that allowed by arbitrary causal theories.

IV. CORRELATIONS ALLOWED BY QUANTUM MECHANICS

Although the necessary and sufficient conditions defining the set of quantum correlations \(Q\) in the EPR-Bell scenario have long been known [30], surprisingly they are rarely mentioned in the literature. In contrast, some necessary (but not sufficient) conditions for a set of four correlations to belong to \(Q\) are much better known. Let us start by reviewing the two most famous ones.

A. Tsirelson’s linear inequalities

Tsirelson [7] showed that, for any quantum state \(\rho\), the four quantum correlations \(\langle A_jB_j \rangle\), for \(i,j = 0,1\), must satisfy eight linear inequalities (usually referred to as Tsirelson’s inequalities, but here we shall call them Tsirelson’s linear inequalities), which can be written as

\[
|\langle A_0B_0 \rangle + \langle A_0B_1 \rangle + \langle A_1B_0 \rangle + \langle A_1B_1 \rangle - 2\langle A_jB_j \rangle| \leq 2\sqrt{2}, \quad (6)
\]

for all \(i,j = 0,1\) [29]. Different proofs of the inequalities (6) can be found in [31–33]. Quantum mechanics predicts violations of the CHSH inequalities (2) up to \(2\sqrt{2}\). Such violations can be obtained with pure [2] or mixed states [34]. A method for deriving maximal violations of Bell-type inequalities can be found in [35].

The volume of the set \(T\) defined by Tsirelson’s linear inequalities (6) is

\[
V_T = 0.961 \times 2^4. \quad (7)
\]

B. Uffink’s quadratic inequalities

Uffink’s quadratic inequalities [23] provide a more restrictive necessary (but still not sufficient) condition for the correlations to be attainable by quantum mechanics. According to Uffink, the four correlations must satisfy the following inequalities:

\[
\langle A_0B_0 \rangle + \langle A_0B_1 \rangle + \langle A_1B_0 \rangle + \langle A_1B_1 \rangle - 2\langle A_jB_j \rangle \leq 2\sqrt{2}, \quad (6)
\]

\[
\langle A_0B_0 \rangle - \langle A_0B_1 \rangle + \langle A_1B_0 \rangle + \langle A_1B_1 \rangle - 2\langle A_jB_j \rangle \leq 2\sqrt{2}, \quad (7)
\]

\[
\langle A_0B_0 \rangle + \langle A_0B_1 \rangle - \langle A_1B_0 \rangle - \langle A_1B_1 \rangle - 2\langle A_jB_j \rangle \leq 2\sqrt{2}, \quad (8)
\]

\[
\langle A_0B_0 \rangle - \langle A_0B_1 \rangle - \langle A_1B_0 \rangle + \langle A_1B_1 \rangle - 2\langle A_jB_j \rangle \leq 2\sqrt{2}. \quad (9)
\]

C. Tsirelson’s and Landau’s inequalities

However, both Tsirelson’s linear inequalities (6) and Uffink’s quadratic inequalities (8) are necessary but not sufficient conditions for the correlations to be attainable by local measurements on subsystems of a composite physical system prepared in a quantum state. Although rarely mentioned in the literature, to my knowledge, there are three equivalent sets of necessary and sufficient conditions to define the set \(Q\) of correlations attainable by quantum mechanics. The first was provided by Tsirelson [30]. According to Tsirelson, a set of four correlations \(\langle A_jB_j \rangle\) \((i,j = 0,1)\) is realizable in quantum mechanics (i.e., belongs to \(Q\)) if at least one of the following two inequalities holds:

\[
0 \leq (\langle A_0B_1 \rangle - \langle A_1B_0 \rangle) \times (\langle A_0B_0 \rangle - \langle A_1B_1 \rangle) - \langle A_0B_0 \rangle \times (\langle A_0B_0 \rangle - \langle A_1B_1 \rangle) - \langle A_1B_0 \rangle \times (\langle A_0B_0 \rangle - \langle A_1B_1 \rangle)
\]

\[
\leq 1 - 4\sum_{i,j} \langle A_jB_j \rangle^2 - 2\sum_{i,j} \langle A_jB_j \rangle^4 - 2\prod_{i,j} \langle A_jB_j \rangle, \quad (10)
\]

\[
0 \leq 2\max_{i,j} \langle A_jB_j \rangle^4 - (\max_{i,j} \langle A_jB_j \rangle^2)^2 \sum_{i,j} \langle A_jB_j \rangle^2 + 2\prod_{i,j} \langle A_jB_j \rangle. \quad (11)
\]

The second characterization of \(Q\) is due to Landau [36]. According to him, four correlations belong to \(Q\) if and only if they satisfy the following inequalities:

\[
|\langle A_0B_0 \rangle + \langle A_1B_1 \rangle - \langle A_0B_1 \rangle - \langle A_1B_0 \rangle| \leq \sqrt{1 - \langle A_0B_0 \rangle^2 \sqrt{1 - \langle A_1B_1 \rangle^2}}
\]

\[
+ \sqrt{1 - \langle A_0B_0 \rangle^2 \sqrt{1 - \langle A_1B_1 \rangle^2}}. \quad (12)
\]

These inequalities (12) are equivalent to inequalities (10) and (11) [29].

The third equivalent definition of \(Q\) can be explicitly found for the first time in [29] (although it can be easily derived from the results in [36]). According to this, four correlations belong to \(Q\) if and only if they satisfy the following eight inequalities:

\[
|\arcsin(A_0B_0) + \arcsin(A_0B_1) + \arcsin(A_1B_0) + \arcsin(A_1B_1) - 2\arcsin(A_jB_j)| \leq \pi, \quad (13)
\]

for all \(i,j = 0,1\). Using inequalities (13) to describe \(Q\) has the advantage of being analogous to using inequalities (2) to describe \(C\). These inequalities (13) have been recently rediscovered by Musanes [37] (see also [38]).

D. Main results

The simplest way to calculate the volume of \(Q\), which is a four-dimensional convex set [29], is by using expression (12). Then, it can be seen that
\[ V_Q = \frac{3\pi^2}{2} \approx 0.925 \times 2^4. \]  

(14)

Therefore the ratio between the volumes of the set of quantum correlations and those allowed by local hidden-variable theories, which, as explained in Sec. I A is a natural measure of how much larger than classical correlations quantum correlations are for the EPR-Bell scenario, is

\[ \frac{V_Q}{V_C} = \frac{\left(\frac{3\pi}{8}\right)^2}{32} \approx 1.388. \]  

(15)

Although the number \( \pi \) is ubiquitous in nature [39,40], it is very surprising to find it again in this context.

On the other hand, the ratio between the volumes of the set of quantum correlations and those allowed by arbitrary causal theories is

\[ \frac{V_Q}{V_C} = \frac{3\pi^2}{32} \approx 0.925. \]  

(16)

Popescu and Rohrlich’s question was why quantum correlations do not violate the CHSH-Bell inequalities “more” than they do [8]. Result (16) allows us to quantify how much larger than the set of quantum correlations the set of possible correlations could be: 7.5% of the, in principle, possible sets of four correlations never occur in nature.

In addition, we can use \( V_Q \) to quantify the success of previous characterizations of \( Q \). For instance, comparing \( V_T \), given by Eq. (7), with \( V_Q \), given by Eq. (14), we obtain that the set \( T \) defined by Tsirelson’s linear inequalities (6) is 3.8% larger than \( Q \) (i.e., 3.7% of the sets of four correlations belonging to \( T \) are not actually achievable by quantum mechanics). On the other hand, comparing \( V_U \), given by Eq. (9), with \( V_Q \), we obtain that the set \( U \) defined by Uffink’s linear inequalities (8) is 2.6% larger than \( Q \) (i.e., 2.6% of the sets of four correlations belonging to \( T \) are not actually achievable by quantum mechanics).

V. FURTHER LINES OF RESEARCH

In this paper we have investigated the basic scenario where quantum correlations are different than local ones: \( N = 2 \) parties, each of them choosing between \( M = 2 \) alternative local experiments, each of which have \( D = 2 \) possible outcomes. There are two basic lines to extend these results:

One line for future research is to investigate how the ratios \( V_Q/V_C \) and \( V_Q/V_C \) evolve in more complex scenarios, namely, those involving more parties, more alternative local experiments per party, and more outcomes per experiment. In particular, it would be interesting to know how \( V_Q/V_C \) evolves with \( N \), assuming that \( M = 2 \) and \( D = 2 \) are fixed. For this case, Mermin showed a Bell inequality which is violated by quantum predictions by an amount that grows exponentially with \( N \) [41]. Does this happen with \( V_Q/V_C \) or, on the contrary, does \( V_Q/V_C \) decrease with \( N \) and is there a “classical limit”? The analytical expression of \( V_Q/V_C \) as a function of \( N \) would require us to know what are the necessary and sufficient conditions which define the set of quantum correlations for these more complex scenarios, something which is still an open problem. However, some numerical research can be attempted.

On the other hand, we have just paid attention to the space of correlations, which is just a four-dimensional projection of the (eight-dimensional, if we assume no-signaling) space of joint probabilities. In general, there are many different sets of joint probabilities giving the same set of correlations. For instance, in the EPR-Bell scenario, the fact that four numbers satisfy Eq. (13) only implies that there exists at least one compatible set of eight joint probability distributions obtainable by performing measurements on a quantum state, but it does not mean that any possible set of eight joint probabilities satisfying Eq. (13) is obtainable by performing measurements on a quantum state. Therefore another interesting line for future research would be calculating the volume of the set of quantum joint probabilities and comparing it with the volume of the joint probabilities allowed by local hidden-variable theories. The former could be calculated numerically, since the necessary and sufficient condition for eight probabilities to be allowed by quantum mechanics can be expressed as a “Russian doll”-type set of theorems [7] that can be implemented as a computer program. However, the analytical solution could be much more difficult to obtain, since it still is an open problem whether or not it is possible to define the set of quantum joint probabilities by means of a set of polynomial, or even analytical, inequalities [29].

VI. CONCLUSIONS

How much “larger” than classical correlations are quantum correlations? We have shown that the volume of the set of physically obtainable correlations \( \langle A_iB_j \rangle \) in the EPR-Bell scenario, namely, where \( A_0 \) and \( A_1 \) (\( B_0 \) and \( B_1 \)) are two alternative dichotomic experiments on subsystem \( A \) (on a distant subsystem \( B \)), assumed to be those obtainable by local measurements on quantum states, is \( (3\pi/8)^2 \approx 1.388 \) larger than the volume of the set of correlations obtainable by local hidden-variable theories.

How much “larger” could, in principle, the set of correlations be? We have shown that the set of quantum correlations is only \( 3\pi^2/32 = 0.925 \) of the volume allowed by arbitrary causal (i.e., no-signaling) theories.

In other words, given four real random numbers between −1 and +1, the probability for them to be reproducible by local hidden-variable theories is 2/3. The probability for them to be physically obtainable (i.e., reproducible by quantum mechanics) is \( 3\pi^2/32 = 0.925 \). The probability for them to be reachable by arbitrary causal theories is 1.

In addition, we have shown that the sets defined by Tsirelson’s linear inequalities [7] and Uffink’s quadratic inequalities [23] contain 3.7% and 2.6%, respectively, of elements that cannot be obtained within quantum mechanics.

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HOW MUCH LARGER QUANTUM CORRELATIONS ARE...